Approximability of the Two-Stage Knapsack problem with discretely distributed weights

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1 The Two-Stage Knapsack Problem

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Outline

The Deterministic Knapsack Problem

- $c > 0$: Knapsack weight capacity
- n items
- $r_i > 0$: reward of item i
- w_i: weight of item i

Objective

Maximize the total reward of chosen items whose total weight respect knapsack capacity.

Applications

Logistics - Resource allocation - Scheduling - Network Optimization etc.

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The **Stochastic** Knapsack Problem **with Random Weights**

- $c > 0$: Knapsack weight capacity
- n items
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- χ_i: **random** weight of item i

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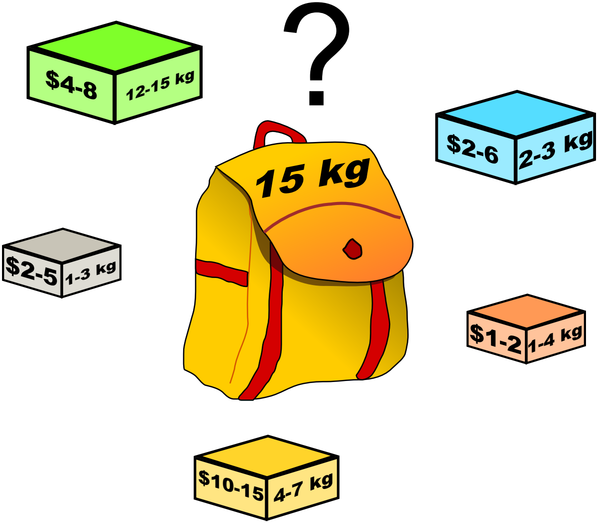
Maximize the total reward of chosen items whose total weight respect knapsack capacity.

Question

How to handle the fact that chosen items might not respect knapsack capacity?


$4-8
12-15 kg
?
$2-6
2-3 kg
15 kg
$2-5
1-3 kg
$1-2
1-4 kg
$10-15
4-7 kg

Two-Stage Setting

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- Second stage: The decision can/has to be corrected

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  ...have to be removed in case of an overweight
  ...can be added if capacity sufficient
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- $\mathbb{P}\{\chi = \chi^k\} = p^k$

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- Weight capacity $\simeq$ Total number of beds
- Items $\simeq$ Travel groups
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  $\rightarrow$ groups have to be relocated in other hotels
- Vacant beds filled with last minute offers

Two-Stage Knapsack Problem

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$$(TSKP) \quad \max_{x \in \{0,1\}^n} \sum_{i=1}^{n} r_i x_i$$

s.t.

x: decision vector of 1st stage

Two-Stage Knapsack Problem

$$(TSKP) \quad \max_{x \in \{0,1\}^n} \sum_{i=1}^{n} r_i x_i + \mathbb{E}[\mathcal{Q}(\mathbf{x}, \chi)]$$

$$\text{s.t.} \quad \mathcal{Q}(x, \chi) = \max_{\mathbf{y}^+, \mathbf{y}^- \in \{0,1\}^n} \sum_{\mathbf{i=1}}^{\mathbf{n}} \bar{\mathbf{r}}_\mathbf{i} \mathbf{y}_\mathbf{i}^+ - \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{d}_\mathbf{i} \mathbf{y}_\mathbf{i}^-,$$

x: decision vector of 1st stage
$\mathbf{y}^+, \mathbf{y}^-$: decision vectors of 2nd stage (recourse action)
$\bar{\mathbf{r}}_\mathbf{i} < \mathbf{r}_\mathbf{i}$, $\mathbf{d}_\mathbf{i} > \mathbf{r}_\mathbf{i}$

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$$(TSKP) \quad \max_{x\in\{0,1\}^n} \sum_{i=1}^{n} r_i x_i + \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{K}} \mathbf{p}^{\mathbf{k}} \mathcal{Q}(\mathbf{x}, \chi^{\mathbf{k}})$$

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Deterministic Equivalent Two-Stage Knapsack Problem

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Deterministic Equivalent Two-Stage Knapsack Problem

$$
\begin{aligned}
(TSK^D) \quad \max \quad & \sum_{i=1}^{n} r_i x_i + \sum_{k=1}^{K} p^k \left(\sum_{i=1}^{n} \overline{r}_i (\mathbf{y}^+)_\mathbf{i}^\mathbf{k} - \sum_{i=1}^{n} d_i (\mathbf{y}^-)_\mathbf{i}^\mathbf{k} \right) \\
\text{s.t.} \quad & (\mathbf{y}^+)_\mathbf{j}^\mathbf{k} \leq 1 - x_j && j = 1, \dots, n, \ \mathbf{k} = \mathbf{1}, \dots, \mathbf{K}, \\
& (\mathbf{y}^-)_\mathbf{j}^\mathbf{k} \leq x_j && j = 1, \dots, n, \ \mathbf{k} = \mathbf{1}, \dots, \mathbf{K}, \\
& \sum_{i=1}^{n} (x_i + (\mathbf{y}^+)_\mathbf{i}^\mathbf{k} - (\mathbf{y}^-)_\mathbf{i}^\mathbf{k}) \chi_i^k \leq c && \mathbf{k} = \mathbf{1}, \dots, \mathbf{K}, \\
& x \in \{0,1\}^n, \\
& (\mathbf{y}^+)^\mathbf{k}, (\mathbf{y}^-)^\mathbf{k} \in \{0,1\}^n && \mathbf{k} = \mathbf{1}, \dots, \mathbf{K}.
\end{aligned}
$$

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Deterministic Equivalent Two-Stage Knapsack Problem

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\text{s.t.} \quad & (\mathbf{y}^+)_{\mathbf{j}}^{\mathbf{k}} \leq 1 - x_j \qquad j = 1, \ldots, n, \ \mathbf{k} = \mathbf{1}, \ldots, \mathbf{K}, \\
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x: decision vector of 1^{st} stage
$(y^+)^k, (y^-)^k$: decision vectors in scenario k

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Deterministic Reformulation...

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- ...has linear objective and constraints.

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- ...has $(2n+1)K$ constraints.

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- ...has linear objective and constraints.
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- ...has $(2K+1)n$ binary decision variables.
- ...can have **"exponential size"**.
- ...in general intractable with exact solvers.

Published and Working Papers treating the TSK^D

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- S. Kosuch (2011): **Towards an Ant Colony Optimization algorithm for the Two-Stage Knapsack problem**, *Proc. of the VII. ALIO/EURO Workshop on Applied Combinatorial Optimization*

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Outline

Theorem (Main Non-Approximability Result)

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For all $\epsilon > 0$, there exists no $K^{-\frac{1}{2}+\epsilon}$-approximation algorithm for the TSKP, unless $\mathcal{P} = \mathcal{NP}$.

Definition

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We denote *AddTSKP* ($AddTSK^D$) the variant of *TSKP* ($AddTSK^D$) where in the second stage items can only be added.

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- First-stage decision $\rightarrow$ infeasible second stage problem $\Rightarrow$ $f(x) = -\infty$
- Optimal solution of *AddTSKP* always respects capacity

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Theorem (Z. Li'ang & Z. Yin, 1999)

For any $\epsilon > 0$, the multiply-constrained knapsack problem does not admit a $m^{-\frac{1}{4}+\epsilon}$-approximation algorithm unless $\mathcal{P} = \mathcal{NP}$.

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Corollary (Z. Li'ang & Z. Yin, 1999 + D. Zuckerman, 2006)

For any $\epsilon > 0$, the multiply-constrained knapsack problem does not admit a $m^{-\frac{1}{2}+\epsilon}$-approximation algorithm unless $\mathcal{P} = \mathcal{NP}$.

Theorem (Main Non-Approximability Result)

For all $\epsilon > 0$, there exists no $K^{-\frac{1}{2}+\epsilon}$-approximation algorithm for the TSKP, unless $\mathcal{P} = \mathcal{NP}$.

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Schema of the theorem's proof

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Schema of the theorem's proof

- Reduction from the *MCKP*

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- Reduction from the *MCKP*
- $MCKP \quad \leftarrow \quad AddTSK^D \quad \leftarrow \quad TSK^D$

Reformulate an instance of $AddTSK^D$ via an instance of TSK^D (idea)

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Reformulate an instance of $AddTSK^D$ via an instance of TSK^D (idea)

- Given instance $\mathcal{I} = (r, \bar{r}, p, w, c)$ of $AddTSK^D$.
- Construct instance $\mathcal{I}' = (r, \bar{r}, d, p, w, c)$ of TSK^D s.t.

Reformulate an instance of $AddTSK^D$ via an instance of TSK^D (idea)

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- Construct instance $\mathcal{I}' = (r, \bar{r}, d, p, w, c)$ of TSK^D s.t.

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$$\Rightarrow \sum_{j=1}^{m} \frac{1}{m} \sum_{i=1}^{n} \frac{1}{n+1} < 1$$

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Outline

The *TSKP* with linearly dependent first- and second-stage reward vectors

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Lemma

Let $\alpha \in (0,1)$ *and denote TSKP*$(\alpha, \cdot)$ *the variant of the TSKP such that* $\bar{r} = \alpha \cdot r$. *Then there exists an approximation algorithm for the TSKP*$(\alpha, \cdot)$ *with approximation ratio* α.

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Add no item in the first stage.

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Proposition (Genaralisation of the lemma)

For any instance of the TSKP define $\alpha := \min_{i=1\ldots,n} \frac{\bar{r}_i}{r_i}$. Then adding no items in the first stage always yields a solution whose solution value is at least an α-fraction of the optimal solution value.

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If the item weights are independently distributed with polynomial number of realizations, the TSKP admits a $\frac{1}{n}$-approximation algorithm.

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- $\forall k$, solve *KP* $(\bar{r}, w^k, c)$ with *FPTAS*.
- Either output solution of former, or 0.

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- Approximation algorithms for special cases:
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  - → Polynomial scenario model
  - → Fixed number of scenarios ($K - AddTSKP$)

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Future Work

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- $PTAS$ for $K-(Add)TSKP$
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Future Work

- More complex approximation algorithms for special cases
- *PTAS* for $K-(Add)TSKP$
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- Approximation in case of continuous distributions

Thank you!

:-)

Grazie!

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