On a stochastic knapsack problem

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1 Introduction

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2. Chance-Constrained Knapsack Problem
   - Problem Formulation

The Deterministic Knapsack Problem

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Problem

How to handle possibility of violated capacity constraint?

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Expectation-Constrained Knapsack Problem (ECKP)

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- Apply "Integration by parts"-method to obtain needed gradient

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- Example: normal distribution

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  - use $\nabla_x \ell(x^{k-1}, \lambda^{k-1}, \hat{\chi})$ (instead of $\nabla_x \mathcal{L}(x^{k-1}, \lambda^{k-1})$) to update x^k

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For all $i = 1, ..., n$: If $x_i^k > 1$ set $x_i^k = 1$ / If $x_i^k < 0$ set $x_i^k = 0$
If $\lambda_i^k > 0$ set $\lambda_i^k = 0$

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- $\Rightarrow \ell$ non differentiable

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- replace $\nabla_x \ell(x^{k-1},\lambda^{k-1},\chi^k)$ by $\nabla_x \widetilde{\ell}(x^{k-1},\lambda^{k-1},\chi^k)$ in stochastic gradient algorithm

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Integration by Parts method $\rightarrow$ h-th component of the gradient:

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Numerical Problem 2: Slow convergence

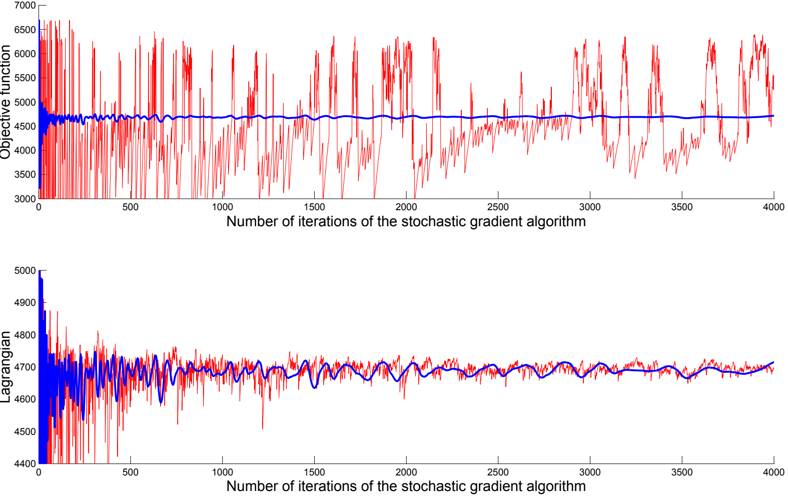
$$\left(\nabla_x \widetilde{\ell}(x,\lambda,\chi)\right)_h =$$

$$r_h\chi_h - \lambda\left(\mathbb{1}_{\mathbb{R}^+}(\sum_{i=1}^{n}\chi_i x_i - c)\frac{(\chi_\kappa - \mu_\kappa)}{x_\kappa \sigma_\kappa^2}\left(\chi_h - \frac{(\sum_{i=1}^{n}\chi_i x_i - c)}{x_\kappa}\nu_h^\kappa\right)\right)$$

Solution: Choose κ wisely

- in iter. k: $\kappa \in \{1,\ldots,n\}$ (freely) chosen s.t. $x_\kappa^k > 0$
- Remember: feasible set $X = \{x \in [0,1]^n | \; ||x||_\infty \geq \frac{1}{n}\}$
  $\Rightarrow$ such κ exists if x^k feasible
- choose κ (if possible) as

$$\kappa = \arg\max_{i=1,\ldots,n}\{x_i^k | x_i^k \geq \frac{1}{n} \wedge (\chi_i^k - \mu_i) > 0\}$$


Objective function
7000
6500
6000
5500
5000
4500
4000
3500
3000
0
500
1000
1500
2000
2500
3000
3500
4000
Number of iterations of the stochastic gradient algorithm
Lagrangian
5000
4900
4800
4700
4600
4500
4400
0
500
1000
1500
2000
2500
3000
3500
4000
Number of iterations of the stochastic gradient algorithm

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CDF involved

$$\mathbb{E}\left[\mathbb{1}_{\mathbb{R}^+}(c - \sum_{i=1}^n \chi_i x_i)\right] = \mathcal{P}\{\sum_{i=1}^n \chi_i x_i \leq c\} = \Phi(c)$$

where Φ is the cumulative distribution function of $\sum_{i=1}^n \chi_i x_i$

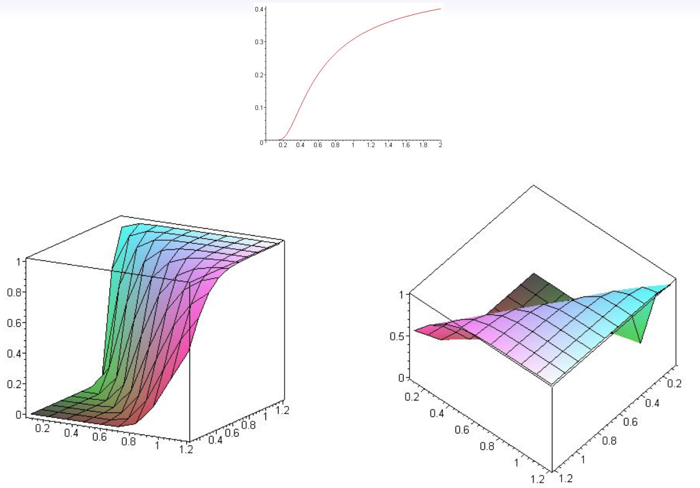

Figure: Constraint function in 1 and 2-dimensional case with realistic constants

Outline

	Arrow-Hurwicz & IP-method		
n	**considered nodes**	**CPU-time (sec) B-and-B**	**Gap**
15	34	0.06	0 %
20	63	0.13	0 %
30	436	1.22	0 %
50	7051	31.34	0 %
75	23911	161.47	0 %
100	98479	1049.18	?
150	*	*	*

* CPU-time exceeds 1h

Table: Numerical results for the (combinatorial) *ECKP*

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- Convergence Issues $\rightarrow$ solved with **more intelligent choices**
- Able to solve (Binary) Chance-Constrained Knapsack problem with up to 100 items in less than $1h$

Thank you!

Danke!