Stochastic Optimization
IDA PhD course 2011ht

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6. Lecture: Two- and Multi-Stage Problems
17. November 2011

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1 Two-Stage Programming
- Deterministic reformulation in case of discrete distributions
- Main theoretical result

Outline

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Why Two-Stage Programming?

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- Feasibility in all scenarios: Conservative solutions
- Chance-Constraints: Infeasibility of solution possible
- Simple Recourse: No actual "recourse" action

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- Feasibility in all scenarios: Conservative solutions
- Chance-Constraints: Infeasibility of solution possible
- Simple Recourse: No actual "recourse" action
- In the end: feasible solution $\rightarrow$ most realistic?

Basic structure of Two-Stage Programming Problem

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1. First stage: random parameters unknown
2. (Pre)-decision taken based on (statistical) information of uncertain parameters
3. Objective: Minimize expected total cost
4. Between first and second stage: **all** random parameters come to be known
5. Second stage: deterministic problem
6. Make corrective decision ("recourse"):
   - $\rightarrow$ Make solution feasible
   - $\rightarrow$ Decrease total cost

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Linear Two-Stage Problem

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$$\min_{x \geq 0} \quad c^T x$$

s.t.

$x \in \mathbb{R}^{n_1}$: decision vector of 1$^{\text{st}}$ stage

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$$\begin{aligned} \min_{x \geq 0} \quad & c^T x + \mathbb{E}[\mathcal{Q}(x,\chi)] \\ \text{s.t.} \quad & Ax \geq b, \\ & \mathcal{Q}(x,\chi) = \min_{y \geq 0} \quad d^T y \end{aligned}$$

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1. Two-Stage Programming
   - Deterministic reformulation in case of discrete distributions
   - Main theoretical result

2. Multi-Stage Programming
   - Comparison with Online Optimization
   - Modeling Multi-Stage Stochastic Programming Problems
   - Deterministic Reformulation in case of discrete distributions (Main Idea)

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Deterministically Reformulated Linear Two-Stage Problem

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A Linear Two Stage Programming Problem is said to have **fixed recourse** if the recourse matrix is deterministic, i.e. $W(\chi) = W$.

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Basics of Multi-Stage Programming

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- Crucial: Discretization of time and random distribution(s)

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- Possible objectives:
  - a) Minimize **total cost**
  - b) Average cost not higher than r times "optimal" cost

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Similarities

Similarities

- Decision taken over time (stages $\leftrightarrow$ periods)

Similarities

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- Decision does not depend on information available in future

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Differences

Solution

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Differences

Solution

Multi-Stage Programming:

Differences

Solution

Multi-Stage Programming:

- Decision vector

Differences

Solution

Multi-Stage Programming:

- Decision vector
- Decision for each stage and scenario

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Online Optimization:

Differences

Solution

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- Decision vector
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Online Optimization:

- Policy

Differences

Solution

Multi-Stage Programming:

- Decision vector
- Decision for each stage and scenario

Online Optimization:

- Policy
- Online Algorithm

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Differences

Objective

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Multi-Stage Programming:

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Differences

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Multi-Stage Programming:

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Online Optimization:

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Differences

Objective

Multi-Stage Programming:

- Minimize expected total cost

Online Optimization:

- Minimize total cost
- Propose "competitive" online algorithm

Differences

Modeling

Differences

Modeling

Multi-Stage Programming:

Differences

Modeling

Multi-Stage Programming:

- Random variables

Differences

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- Knowledge about distribution

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Online Optimization:

Differences

Modeling

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Online Optimization:

- Generally no knowledge of structure of input

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Differences

Modeling

Multi-Stage Programming:

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Online Optimization:

- Generally no knowledge of structure of input
- State sequence

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Differences

Modeling

Multi-Stage Programming:

- Random variables
- Knowledge about distribution

Online Optimization:

- Generally no knowledge of structure of input
- State sequence
- Deterministic formulation

Two different views

Two different views

Multi-Stage programming is...

Two different views

Multi-Stage programming is...

- ... a different approach to optimization with information over time than...

...online optimization.

Two different views

Multi-Stage programming is...

- ... a different approach to optimization with information over time than...
- ... one way to attack problems in the domain of...

...online optimization.

Which approach to choose?

Questions to be answered:

- Do I have a finite, infinite or undefined number of periods/stages?

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  OR: Do I need/want to learn about uncertainties "on the fly"?

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  OR: Do I need/want to learn about uncertainties "on the fly"?
- Is discretization of the random distribution an option?
- Do I have the infrastructure and/or time to make decisions online?
- Will I use my solution once or several times?

Outline

General Linear Multi-Stage Model

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General Linear Multi-Stage Model

$$\min_{x_0 \geq 0} \quad c_0^T x_0$$

s.t.

General Linear Multi-Stage Model

$$\begin{aligned} \min_{x_0 \geq 0} \quad & c_0^\top x_0 \\ \text{s.t.} \quad & A_{00} x_0 \geq b_0, \end{aligned}$$

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General Linear Multi-Stage Model

$$\min_{x_0 \geq 0} \quad c_0^\intercal x_0 + \mathbb{E}_{\zeta_1}\left[\mathcal{Q}^1(x_0, \zeta_1)\right]$$

$$\text{s.t.} \quad A_{00} x_0 \geq b_0,$$

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$$\mathcal{Q}^1(x_0,\zeta_1) = \min_{x_1 \geq 0} \quad c_1^\intercal x_1$$

$x_1 = x_1(\zeta_1)\ \forall t \geq 1$

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$$\mathcal{Q}^1(x_0, \zeta_1) = \min_{x_1 \geq 0} \quad c_1^\intercal x_1$$

$$\text{s.t.} \quad \sum_{s=0}^{1} A_{1s}(\zeta_1) x_s \geq b_1(\zeta_1).$$

$x_1 = x_1(\zeta_1) \; \forall t \geq 1$

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General Linear Multi-Stage Model

$$\min_{x_0 \geq 0} \quad c_0^\intercal x_0 + \mathbb{E}_{\zeta_1}\left[\mathcal{Q}^1(x_{[0]}, \zeta_{[1]})\right]$$

$$\text{s.t.} \quad A_{00}x_0 \geq b_0,$$

$$\mathcal{Q}^T(x_{[T-1]}, \zeta_{[T]}) = \min_{x_T \geq 0} \quad c_T^\intercal x_T$$

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$\zeta_{[t]} = (\zeta_1, \ldots, \zeta_t)$
$x_{[t-1]} = (x_0, x_1, \ldots, x_{t-1})$
$x_t = x_t(\zeta_{[t]}) \; \forall t \geq 1$

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General Linear Multi-Stage Model

$$
\begin{aligned}
\min_{x_0 \geq 0} \quad & c_0^\intercal x_0 + \mathbb{E}_{\zeta_1}\left[\mathcal{Q}^1(x_{[0]}, \zeta_{[1]})\right] \\
\text{s.t.} \quad & A_{00} x_0 \geq b_0, \\
& \left.\begin{aligned}
\mathcal{Q}^t(x_{[t-1]}, \zeta_{[t]}) = \min_{x_t \geq 0} \quad & c_t^\intercal x_t + \mathbb{E}_{\zeta_{t+1}|\zeta_{[t]}}\left[\mathcal{Q}^{t+1}(x_{[t]}, \zeta_{[t+1]})\right] \\
\text{s.t.} \quad & \sum_{s=0}^{t} A_{ts}(\zeta_{[t]}) x_s \geq b_t(\zeta_{[t]}).
\end{aligned}\right\} \quad \begin{matrix}\forall t \in \\ \{1, \ldots, t-1\}\end{matrix} \\
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Outline

Definition

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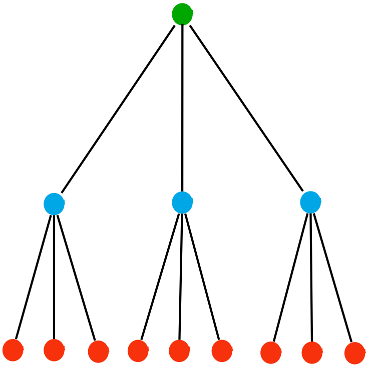
Scenario Tree



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Introduce sequence of decision vectors $(x_0^k, x_1^k, \ldots, x_T^k)$ ($\forall$ scenarios $\zeta_{[T]}^k$)

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Problem

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Solution

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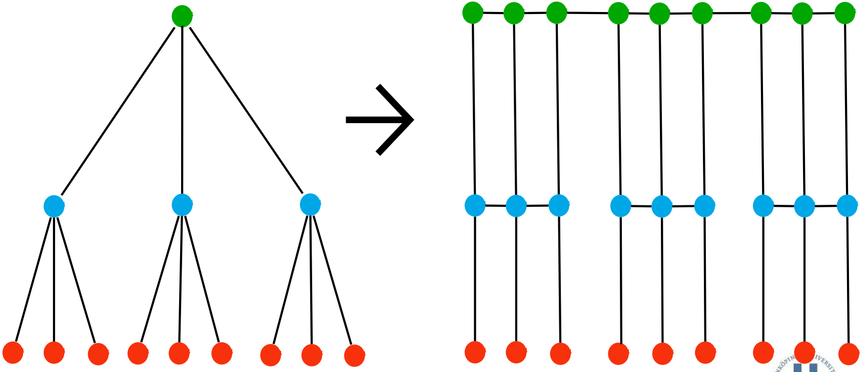

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QUESTIONS?

What about next week?