Stochastic Optimization
IDA PhD course 2011ht

Stefanie Kosuch
PostDok at TCSLab
www.kosuch.eu/stefanie/

5. Lecture: Two-Stage Problems
10. November 2011

1 Randomness occurs in the constraint function

2 Simple Recourse problems - Useful Theorems

3 Two-Stage Programming
- Example: TSKP
- Simple Recourse as a special case of Two-Stage problems
- Deterministic reformulation in case of discrete distributions
- Main theoretical result

Linköping University

Outline

1 Randomness occurs in the constraint function

2 Simple Recourse problems - Useful Theorems

3 Two-Stage Programming
- Example: TSKP
- Simple Recourse as a special case of Two-Stage problems
- Deterministic reformulation in case of discrete distributions
- Main theoretical result

Deterministic Opt. Model $\rightarrow$ Stochastic Programming Model

$$\begin{aligned} \max_{x \in X} \quad & f(x) \\ \text{s.t.} \quad & G(x) \leq 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} \min_{x \in X} \quad & f(x) \\ \text{s.t.} \quad & G(x, \chi) \leq 0 \end{aligned}$$

$\chi \in \Omega \subseteq \mathbb{R}^s$: random vector
$G : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^m$
$G(x, \chi) = (g_1(x, \chi), \ldots, g_m(x, \chi))$

Question

What means "feasible solution" in case of uncertain parameters in the constraint function(s)?

- Feasible in all possible cases → conservative/infeasible/trivial
- Feasible with high probability → chance-constraint ✓
- Average violation not too bad → simple recourse ✓
- Feasible after correction has been made → multi-stage

Outline

1 Randomness occurs in the constraint function

2 Simple Recourse problems - Useful Theorems

3 Two-Stage Programming
- Example: TSKP
- Simple Recourse as a special case of Two-Stage problems
- Deterministic reformulation in case of discrete distributions
- Main theoretical result

Basic Idea(s)

- Allow violation
- Restrict average "amount of violation"
- Introduce penalty per "violation unit"
- Find good trade-off: cost $\leftrightarrow$ penalty

Linköping University

Simple-Recourse problem

$$\min_{x \in X} \quad f(x) + \sum_{i=1}^{m} d_i \cdot \mathbb{E}\left[[g_i(x,\chi)]^+\right]$$

- $d_i > 0 \ \forall i \in \{1, \ldots, m\}$
- $[x]^+ = max(0, x)$

Theorem

Let

- $\gamma(x) := \mathbb{E}\left[[g(x,\chi)]^+\right]$
- $\mathbb{E}\left[g(x,\chi)\right] < \infty \ \forall x \in \mathbb{R}^n$

Then the following holds:

$$g(\cdot,\hat{\chi}) \text{ is convex } \forall \hat{\chi} \quad \Longrightarrow \quad \gamma \text{ is convex}$$

Theorem

Let

- $\gamma(x) := \mathbb{E}\left[[a(\chi)x - b]^+\right]$
- Φ*: cumulative distribution function of* $a(\chi)x - b$
- $\mathbb{E}\left[a(\chi)x - b\right] < \infty \; \forall x \in \mathbb{R}^n$

Then the following holds:

1. γ *is convex*
2. *For all* $x \in \mathbb{R}^n$ $\gamma(x) < \infty$.
3. γ *is Lipschitz continuous.*
4. γ *is differentiable wherever* Φ *is continuous.*
5. $a(\chi)x - b$ *discretely distributed* $\Rightarrow \gamma$ *is piecewise linear.*

Linköping University

Outline

1 Randomness occurs in the constraint function

2 Simple Recourse problems - Useful Theorems

3 Two-Stage Programming
- Example: TSKP
- Simple Recourse as a special case of Two-Stage problems
- Deterministic reformulation in case of discrete distributions
- Main theoretical result

Why Two-Stage Programming?

- Feasibility in all scenarios: Conservative solutions
- Chance-Constraints: Infeasibility of solution possible
- Simple Recourse: No actual "recourse" action
- In the end: feasible solution $\rightarrow$ most realistic?
- Costs/rewards of recourse action taken into account

Basic structure of Two-Stage Programming Problem

1. First stage: random parameters unknown
2. (Pre)-decision taken based on (statistical) information of uncertain parameters
3. Objective: Minimize expected total cost
4. Between first and second stage: **all** random parameters come to be known
5. Second stage: deterministic problem
6. Make corrective decision ("recourse"):
   - $\rightarrow$ Make solution feasible
   - $\rightarrow$ Decrease total cost

General Two-Stage Problem

$$\begin{aligned} \min_{x \in X} \quad & f^1(x) + \mathbb{E}[\mathcal{Q}(x,\chi)] \\ \text{s.t.} \quad & G^1(x) < 0 \\ & \mathcal{Q}(x,\chi) = \min_{y \in Y} \quad f^2(x,y), \\ & \qquad\qquad\quad \text{s.t.} \quad G^2(x,y,\chi) \leq 0. \end{aligned}$$

$x \in \mathbb{R}^{n_1}$: decision vector of 1st stage
$y \in \mathbb{R}^{n_2}$: decision vectors of 2nd stage (recourse action)

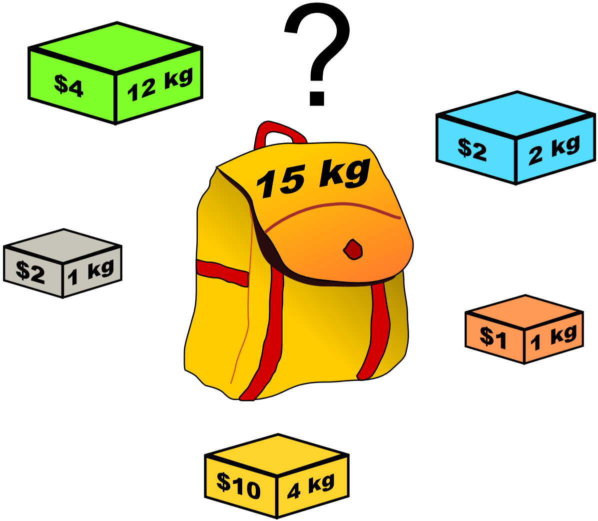
Outline

1 Randomness occurs in the constraint function

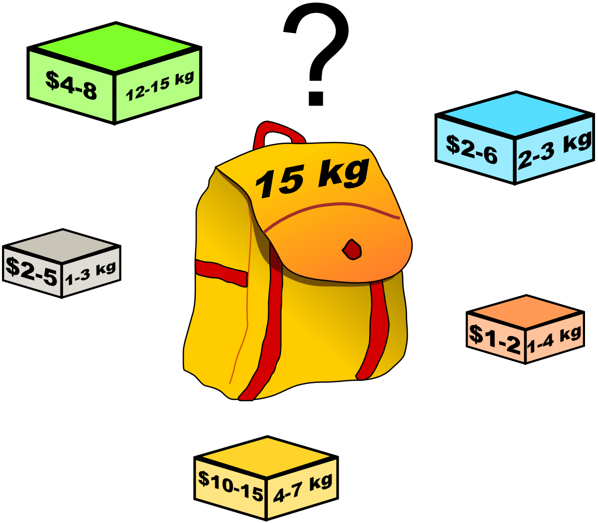
2 Simple Recourse problems - Useful Theorems

3 Two-Stage Programming
- Example: TSKP
- Simple Recourse as a special case of Two-Stage problems
- Deterministic reformulation in case of discrete distributions
- Main theoretical result

Example: TSKP



Example: TSKP



Example: TSKP

Two-Stage Stochastic Knapsack problem

- First stage: items can be put in the knapsack
- First stage $\longleftrightarrow$ second stage: item weights are revealed
- Second stage: Items
  ...have to be removed in case of an overweight
  ...can be added if capacity sufficient
  ...can be exchanged to increase gain.
- Correction of the decision causes penalty

Example: TSKP

Application: Travel Agency

- Knapsack $\simeq$ Hotel Complex
- Weight capacity $\simeq$ Total number of beds
- Items $\simeq$ Travel groups
- Item weights $\simeq$ Group size
- Randomness e.g., cancellations
- Agency allows overbooking
- Number of beds insufficient
  $\rightarrow$ groups have to be relocated in other hotels
- Vacant beds filled with last minute offers

Linköping University

Example: TSKP

Two-Stage Knapsack Problem

$$(TSKP) \quad \max_{x\in\{0,1\}^n} \sum_{i=1}^{n} r_i x_i + \mathbb{E}[\mathcal{Q}(x,\chi)]$$

$$\text{s.t.} \quad \mathcal{Q}(x,\chi) = \max_{\mathbf{y}^+,\mathbf{y}^-\in\{0,1\}^n} \sum_{\mathbf{i}=1}^{\mathbf{n}} \bar{\mathbf{r}}_\mathbf{i}\mathbf{y}_\mathbf{i}^+ - \sum_{\mathbf{i}=1}^{\mathbf{n}} \mathbf{d}_\mathbf{i}\mathbf{y}_\mathbf{i}^-$$

$$\text{s.t.} \quad y_j^+ \leq 1 - x_j \quad (j=1,\ldots,n),$$

$$y_j^- \leq x_j \quad (j=1,\ldots,n),$$

$$\sum_{i=1}^{n} (x_i + y_i^+ - y_i^-)\chi_i \leq c.$$

$x \in \mathbb{R}^n$: decision vector of 1st stage
$\mathbf{y}^+, \mathbf{y}^- \in \mathbb{R}^n$: decision vectors of 2nd stage (recourse action)
$\bar{\mathbf{r}}_\mathbf{i} < \mathbf{r}_\mathbf{i}$, $\mathbf{d}_\mathbf{i} > \mathbf{r}_\mathbf{i}$

Linear Two-Stage Problem

$$\begin{aligned}
\min_{x\geq 0} \quad & c^T x + \mathbb{E}[\mathcal{Q}(x,\chi)] \\
\text{s.t.} \quad & Ax \geq b, \\
& \mathcal{Q}(x,\chi) = \min_{y\geq 0} \quad d^T y \\
& \qquad\qquad\qquad \text{s.t.} \quad T(\chi)x + W(\chi)y \geq h(\chi).
\end{aligned}$$

$x \in \mathbb{R}^{n_1}$: decision vector of 1st stage
$y \in \mathbb{R}^{n_2}$: decision vectors of 2nd stage (recourse action)
$T(\chi)$: Technology matrix
$W(\chi)$: Recourse matrix

Outline

1 Randomness occurs in the constraint function

2 Simple Recourse problems - Useful Theorems

3 Two-Stage Programming
- Example: TSKP
- Simple Recourse as a special case of Two-Stage problems
- Deterministic reformulation in case of discrete distributions
- Main theoretical result

Linear Simple-Recourse problem

$$\min_{x \in X} \quad c^T x + \sum_{i=1}^{m} d_i \cdot \mathbb{E}\left[[h_i(\chi) - T_i(\chi)x]^+ \right]$$

Equivalent formulation in two stages

$$\begin{aligned} \min_{x \geq 0} \quad & c^T x + \mathbb{E}[\mathcal{Q}(x,\chi)] \\ \text{s.t.} \quad & \mathcal{Q}(x,\chi) = \min_{y \geq 0} \quad d^T y \\ & \qquad\qquad\quad \text{s.t.} \quad T(\chi)x + y \geq h(\chi). \end{aligned}$$

$x \in \mathbb{R}^n$: decision vector of 1st stage
$y \in \mathbb{R}^m$: decision vectors of 2nd stage ("violation")

Linear Two-Stage Problem

$$\min_{x \geq 0} \quad c^T x + \mathbb{E}[\mathcal{Q}(x,\chi)]$$

$$\text{s.t.} \quad Ax \geq b,$$

$$\mathcal{Q}(x,\chi) = \min_{y \geq 0} \quad d^T y$$

$$\text{s.t.} \quad T(\chi)x + W(\chi)y \geq h(\chi).$$

$x \in \mathbb{R}^{n_1}$: decision vector of 1st stage
$y \in \mathbb{R}^{n_2}$: decision vectors of 2nd stage (recourse action)

Linear Simple Recourse Problem in two stages

$$\min_{x \geq 0} \quad c^T x + \mathbb{E}[\mathcal{Q}(x,\chi)]$$

$$\text{s.t.} \quad \mathcal{Q}(x,\chi) = \min_{y \geq 0} \quad d^T y$$

$$\text{s.t.} \quad T(\chi)x + \mathbb{I}_n y \geq h(\chi).$$

$x \in \mathbb{R}^n$: decision vector of 1st stage
$y \in \mathbb{R}^m$: decision vectors of 2nd stage ("violation")
$\mathbb{I}_n$: Identity matrix of dimension n

Simple Recourse Problem formulated as Two-Stage Problem

- No first stage constraints.
- $\#$ second stage variables $= \#$ constrains
- $W(\chi) = W = \mathbb{I}_n$

Outline

1 Randomness occurs in the constraint function

2 Simple Recourse problems - Useful Theorems

3 Two-Stage Programming
- Example: TSKP
- Simple Recourse as a special case of Two-Stage problems
- Deterministic reformulation in case of discrete distributions
- Main theoretical result

Linköping University

Linear Two-Stage Problem

$$\min_{x \geq 0} \quad c^T x + \sum_{k=1}^{K} p^k \mathcal{Q}(x, \chi^k)$$

$$\text{s.t.} \quad Ax \geq b,$$

$$\mathcal{Q}(x, \chi^k) = \min_{y^k \geq 0} \quad d^T y^k$$

$$\text{s.t.} \quad T(\chi^k)x + W(\chi^k)y^k \geq h(\chi^k).$$

$$(\forall\, k \in \{1, \ldots, K\})\,.$$

$x \in \mathbb{R}^{n_1}$: decision vector of 1st stage
$y \in \mathbb{R}^{n_2}$: decision vectors of 2nd stage (recourse action)
$\chi^1, \ldots, \chi^K \in \mathbb{R}^s$: scenarios
$\mathbb{P}\{\chi = \chi^k\} := p^k$: probabilities

Linköping University

Deterministically Reformulated Linear Two-Stage Problem

$$\min_{\substack{x\geq 0\\ y^k\geq 0}} \quad c^T x + \sum_{k=1}^{K} p^k \left(d^T y^k\right)$$

$$\text{s.t.} \quad Ax \geq b, \tag{8a}$$

$$T(\chi^k)x + W(\chi^k)y^k \geq h(\chi^k) \quad (\forall\, k \in \{1,\ldots,K\})\,.$$

Outline

1 Randomness occurs in the constraint function

2 Simple Recourse problems - Useful Theorems

3 Two-Stage Programming
- Example: TSKP
- Simple Recourse as a special case of Two-Stage problems
- Deterministic reformulation in case of discrete distributions
- Main theoretical result

Definition (Fixed Recourse)

A Linear Two Stage Programming Problem is said to have **fixed recourse** if the recourse matrix is deterministic, i.e. $W(\chi) = W$.

Linköping University

Theorem

Given: linear two-stage problem
Let:

- *fixed recourse*
- $\{\lambda \in \mathbb{R}^{m_2} : d \geq \lambda W\} \neq \emptyset$
  ($\rightarrow$ second-stage problem primal and dual feasible)

Then $\mathbb{E}\left[\mathcal{Q}(x,\chi)\right]$ *is*

- *real-valued,*
- *piecewise linear and convex in* x*,*
- *Lipschitz continuous in* x*,*
- *sub-differentiable in* x*.*

Linear Two-Stage Problem

$$\begin{aligned} \min_{x\geq 0} \quad & c^T x + \mathbb{E}[\mathcal{Q}(x,\chi)] \\ \text{s.t.} \quad & Ax \geq b, \\ & \mathcal{Q}(x,\chi) = \min_{y\geq 0} \quad d^T y \\ & \qquad\qquad\quad \text{s.t.} \quad T(\chi)x + W(\chi)y \geq h(\chi). \end{aligned}$$

$x \in \mathbb{R}^{n_1}$: decision vector of 1st stage
$y \in \mathbb{R}^{n_2}$: decision vectors of 2nd stage (recourse action)
$T(\chi)$: Technology matrix
$W(\chi)$: Recourse matrix

Theorem

Given: linear two-stage problem
Let:

- *fixed recourse*
- $\{\lambda \in \mathbb{R}^{m_2} : d \geq \lambda W\} \neq \emptyset$
  ($\rightarrow$ second-stage problem primal and dual feasible)

Then $\mathbb{E}\left[\mathcal{Q}(x, \chi)\right]$ *is*

- *real-valued,*
- *piecewise linear and convex in* x*,*
- *Lipschitz continuous in* x*,*
- *sub-differentiable in* x*.*

Theorem

Given: linear two-stage problem
Let:

- *fixed recourse*
- $\{\lambda \in \mathbb{R}^{m_2} : d \geq \lambda W\} \neq \emptyset$
  ($\rightarrow$ second-stage problem primal and dual feasible)
- χ *continuously distributed*

Then $\mathbb{E}\left[\mathcal{Q}(x,\chi)\right]$ *is*

- *real-valued,*
- *piecewise linear and convex in* x*,*
- *Lipschitz continuous in* x*,*
- *continuously differentiable.*

Linköping University

Theorem

Given: linear two-stage problem
Let:

- *fixed recourse*
- $\{\lambda \in \mathbb{R}^{m_2} : d \geq \lambda W\} \neq \emptyset$
  ($\rightarrow$ second-stage problem primal and dual feasible)
- χ *finitely discretely distributed*

Then $\mathbb{E}\left[\mathcal{Q}(x, \chi)\right]$ *is*

- *real-valued,*
- *piecewise linear and convex in* x*,*
- *Lipschitz continuous in* x*,*
- *subdifferentiable in* x*,*
- *a polyhedral convex function.*

Linköping University

QUESTIONS?

What about next week?