Stochastic Optimization
IDA PhD course 2011ht

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4. Lecture: Simple Recourse Problems
03. November 2011

1. Randomness occurs in the constraint function

Outline

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Deterministic Opt. Model $\rightarrow$ Stochastic Programming Model

$$\begin{array}{ll} \max\limits_{x \in X} & f(x) \\ \text{s.t.} & G(x) \leq 0 \end{array} \quad \rightarrow \quad \begin{array}{ll} \min\limits_{x \in X} & f(x) \\ \text{s.t.} & G(x, \chi) \leq 0 \end{array}$$

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$\chi \in \Omega \subseteq \mathbb{R}^s$: random vector
$G : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^m$
$G(x, \chi) = (g_1(x, \chi), \ldots, g_m(x, \chi))$

Question

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Definition (Concave Function)

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A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **concave** if for all $x, y \in dom(f)$ and all $t \in [0, 1]$ it is

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$$

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A function $f : \mathbb{R}^n \to \mathbb{R}$ is log-concave iff for all $x, y \in dom(f)$ and all $\theta \in [0, 1]$ it is

$$f(\theta x + (1 - \theta)y) \geq f(x)^\theta f(y)^{(1-\theta)}$$

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A continuous probability distribution is called a **log-concave probability distribution** if the corresponding density function is log-concave.

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Theorem (Prékopa '72)

Let $g_i(x, y)$ *(*$i = 1, \ldots, m$*) be* **concave functions** *on* $\mathbb{R}^n \times \mathbb{R}^s$ *(where* x *is an n-dimensional and* y *an s-dimensional vector). Let further* χ *be an s-dimensional random vector with* **logarithmic concave probability distribution**. *Then, the left hand side x-function of the joint chance-constraint*

$$\mathbb{P}\{g_i(x, \chi) \geq 0, i = 1, \ldots, r\} \geq p \tag{1}$$

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Let $g_i(x, y)$ *(*$i = 1, \ldots, m$*) be* **concave functions** *on* $\mathbb{R}^n \times \mathbb{R}^s$ *(where* x *is an* n*-dimensional and* y *an* s*-dimensional vector). Let further* χ *be an* s*-dimensional random vector with* **logarithmic concave probability distribution**. *Then the joint chance-constraint*

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Generalization by Tamm ('76/'77)

Prékopa's results stay valid if the g_i's are only quasi-concave!

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Examples of log-concave probability distributions

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Basic Idea(s)

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- Find good trade-off: cost $\leftrightarrow$ penalty

Simple-Recourse problem

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$$\min_{x \in X} \quad f(x)$$

Simple-Recourse problem

$$\min_{x \in X} \quad f(x) + \sum_{i=1}^{m} d_i \cdot \mathbb{E}\left[[g_i(x,\chi)]^+\right]$$

- $d_i > 0 \ \forall i \in \{1, \ldots, m\}$

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- $[x]^+ = max(0, x)$

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Discrete Finite Distribution

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Problems

- ?

g_i linear / Normal Distribution / Independence

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$a^i_j(\chi) \sim \mathcal{N}(\mu^i_j, \sigma^i_j)$

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φ: Density function of standard normal distribution
Φ: Cumulative distribution function of standard normal distribution

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$\mu_x^i := \sum_{j=1}^n \mu_j^i x_j$

$\sigma_x^i := \sqrt{\sum_{j=1}^n (\sigma_j^i)^2 x_j^2}$

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g_i linear / Normal Distribution / Independence II

φ: Density function of standard normal distribution
Φ: Cumulative distribution function of standard normal distribution
$\mu_x^i := \sum_{j=1}^n \mu_j^i x_j$
$\sigma_x^i := \sqrt{\sum_{j=1}^n (\sigma_j^i)^2 x_j^2}$

Deterministically reformulated problem

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Deterministically reformulated problem

$$\min_{x \in X} \quad f(x) + \sum_{i=1}^m d_i \cdot \left[\sigma_x^i \cdot \varphi\left(\frac{b_i - \mu_x^i}{\sigma_x^i}\right) - (b_i - \mu_x^i) \cdot \left[1 - \Phi\left(\frac{b_i - \mu_x^i}{\sigma_x^i}\right)\right]\right]$$

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g_i linear / Normal Distribution / Independence III

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- Objective function evaluation easy.

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Good news

- Objective function evaluation easy.
- f convex $\Longrightarrow$ Objective function convex.

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Problems

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Problems

- No analytic description.

G linear / Poisson Distribution / Independence

Simple-Recourse Problem

G linear / Poisson Distribution / Independence

Simple-Recourse Problem

$$\min_{x \in X} \quad f(x) + d \cdot \mathbb{E}\left[[\chi^T x - b]^+\right]$$

G linear / Poisson Distribution / Independence

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$$\min_{x \in X} \quad f(x) + d \cdot \mathbb{E}\left[[\chi^T x - b]^+\right]$$

$\chi \in \mathbb{R}^n$: random vector with independent Poisson distr. entries

G linear / Poisson Distribution / Independence

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G linear / Poisson Distribution / Independence

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Deterministically reformulated problem

$$\min_{x \in X} \quad f(x) + d \cdot \sum_{k=1}^{\infty} \frac{e^{-\hat{\mu}} \hat{\mu}^k}{k!} [k - b]^+$$

$$\text{s.t.} \quad \hat{\mu} = \mu^T x$$

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G linear / Poisson Distribution / Independence

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Deterministically reformulated problem

$$\min_{x \in X} \quad f(x) + d \cdot \sum_{k=b+1}^{\infty} \frac{e^{-\hat{\mu}} \hat{\mu}^k}{k!} (k - b)$$
$$\text{s.t.} \quad \hat{\mu} = \mu^T x$$

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Outline

Theorem

Let

- $\gamma(x) := \mathbb{E}\left[[g(x,\chi)]^+\right]$

Theorem

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- $\gamma(x) := \mathbb{E}\left[[g(x,\chi)]^+\right]$
- $\mathbb{E}\left[g(x,\chi)\right] < \infty \; \forall x \in \mathbb{R}^n$

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Then the following holds:

$$g(\cdot,\hat{\chi}) \text{ is convex } \forall \hat{\chi} \quad \Longrightarrow \quad \gamma \text{ is convex}$$

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Theorem

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Then the following holds:

1. γ *is convex*
2. *For all* $x \in \mathbb{R}^n$ $\gamma(x) < \infty$.

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Theorem

Let

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Then the following holds:

1. γ *is convex*
2. *For all* $x \in \mathbb{R}^n$ $\gamma(x) < \infty$.
3. γ *is Lipschitz continuous.*

Theorem

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2. *For all* $x \in \mathbb{R}^n$ $\gamma(x) < \infty$.
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4. γ *is differentiable wherever* Φ *is continuous.*

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Theorem

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Then the following holds:

1. γ *is convex*
2. *For all* $x \in \mathbb{R}^n$ $\gamma(x) < \infty$.
3. γ *is Lipschitz continuous.*
4. γ *is differentiable wherever* Φ *is continuous.*
5. $a(\chi)x - b$ *discretely distributed* $\Rightarrow \gamma$ *is piecewise linear.*

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QUESTIONS?

What about next week?