Stochastic Optimization
IDA PhD course 2011ht

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3. Lecture: Chance-Constrained Programming
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1 Randomness occurs in the constraint function

2 Chance-Constrained Stochastic Optimization Problems

3 Deterministic Reformulations (Special cases)

4 Convexity of Chance Constraints
- Definitions
- Main Results by Prékopa
- Generalized Results

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Outline

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Deterministic Opt. Model → Stochastic Programming Model

$$\begin{array}{ll} \max_{x \in X} & f(x) \\ \text{s.t.} & G(x) \leq 0 \end{array} \quad \rightarrow \quad \begin{array}{ll} \min_{x \in X} & f(x) \\ \text{s.t.} & G(x, \chi) \leq 0 \end{array}$$

$\chi \in \Omega \subseteq \mathbb{R}^s$: random vector
$G : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^m$
$G(x, \chi) = (g_1(x, \chi), \ldots, g_m(x, \chi))$

Question

What means "feasible solution" in case of uncertain parameters in the constraint function(s)?

- Feasible in all possible cases / scenarios
- Feasible with high probability
- Average violation not too bad / Penalty not too high
- Feasible after correction has been made

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Chance-Constrained Stochastic Optimization Problem

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \quad \mathbb{P}\{G(x,\chi) \leq 0\} \geq p$$

Lower bound for probability of feasibility

Stochastic Optimization Problem with Joint Chance-Constraint

$$\begin{aligned} \min_{x \in X} \quad & f(x) \\ \text{s.t.} \quad & \mathbb{P}\{g_i(x,\chi) \le 0 \quad \forall\, i \in \{1,\dots,m\}\} \ge p \end{aligned}$$

Stochastic Optimization Problem with Separate Chance-Constraints

$$\begin{aligned} \min_{x \in X} \quad & f(x) \\ \text{s.t.} \quad & \mathbb{P}\{g_i(x,\chi) \le 0\} \ge p \quad \forall\, i \in \{1,\dots,m\} \end{aligned}$$

Reformulation Joint Chance Constraint

Stochastic Optimization Problem with (Joint) Chance-Constraint

$g(x,\chi) = max_{i\in\{1,\dots,m\}} g_i(x,\chi)$
$g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$

$$\begin{aligned} \min_{x\in X} \quad & f(x) \\ \text{s.t.} \quad & \mathbb{P}\{g(x,\chi) \leq 0\} \geq p \end{aligned}$$

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Upper bound for risk of violation

Stochastic Optimization Problem with Joint Chance-Constraint

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \quad \mathbb{P}\{\exists i \in \{1, \ldots, m\} : \quad g_i(x, \chi) > 0\} \leq 1 - p$$

Stochastic Optimization Problem with Separate Chance-Constraints

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \quad \mathbb{P}\{g_i(x, \chi) > 0\} \leq 1 - p \quad \forall\, i \in \{1, \ldots, m\}$$

Chance-Constraint $\rightarrow$ Expectation-Constraint

Expectation-Constrained Stochastic Optimization Problem

$$\begin{aligned} \min_{x\in X} \quad & f(x) \\ \text{s.t.} \quad & \mathbb{E}\left[G(x,\chi)\right] \leq 0 \end{aligned}$$

Chance-Constraint $\rightarrow$ Expectation-Constraint II

Reformulation

$\mathbb{1}_{\mathcal{I}} : \mathbb{R} \rightarrow \{0,1\}$: Indicator function for interval $\mathcal{I}$
$g : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}$

$$\mathbb{P}\{g(x,\chi) \leq 0\} = \mathbb{E}\left[\, \mathbb{1}_{(-\infty,0]}[g(x,\chi)] \,\right]$$

Stochastic Optimization Problem with *Chance-Constraint*

$$\begin{aligned} \min_{x \in X} \quad & f(x) \\ \text{s.t.} \quad & \mathbb{E}\left[\, \mathbb{1}_{(-\infty,0]}[g(x,\chi)] \,\right] \geq p \end{aligned}$$

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Discrete Finite Distribution

Chance-Constrained Stochastic Optimization Problem

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \quad \mathbb{P}\{G(x,\chi) \leq 0\} \geq p$$

$\chi^1, \ldots, \chi^K \in \mathbb{R}^s$: scenarios
$\mathbb{P}\{\chi = \chi^k\} := p^k$: probabilities

Reformulate Problem Deterministically

Basic idea:

- "Choose" scenarios where constraints are satisfied
- Probability that one of these arises at least p

Realization:

- Introduce one binary decision variable z^k per scenario
- $z^k = 1$: $G(x, \chi^k) \leq 0$

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Discrete Finite Distribution II

Reformulate Problem Deterministically

Basic idea:

- "Choose" scenarios where constraints are satisfied
- Probability that one of these arises at least p

Realization:

- Introduce one binary decision variable z^k per scenario
- $z^k = 1$: $G(x, \chi^k) \leq 0$

Deterministically reformulated problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x, \chi^k) \leq 0 + M(1 - z^k) \quad \forall i = 1, \ldots, m, \forall k = 1, \ldots, K \\ & \sum_{k=1}^{K} p^k z^k \geq p \\ & x \in X, \quad z^k \in \{0, 1\} \quad \forall k = 1, \ldots, K \end{aligned}$$

M: some "big" constant

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Discrete Finite Distribution III

Deterministically reformulated problem

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x, \chi^k) \leq 0 + M(1 - z^k) \quad \forall i = 1, \ldots, m \,, \forall k = 1, \ldots, K \\
& \sum_{k=1}^{K} p^k z^k \geq p \\
& x \in X, \quad z^k \in \{0, 1\} \quad \forall k = 1, \ldots, K
\end{aligned}
$$

M: some "big" constant

Problems

- Numerical instability due to big M possible
- $K \cdot m + 1$ constraints
- K additional binary decision variables

G linear / Normal Distribution

Chance-Constrained Stochastic Programming Problem

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \quad \mathbb{P}\{\chi^T x \leq b\} \geq p$$

$\chi \in \mathbb{R}^n$: random vector with normally distr. entries
$\chi \sim \mathcal{N}(\mu, \Sigma)$
Σ: Covariance Matrix of χ

Deterministically reformulated problem

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \ \Phi\left(\frac{b - \mu^T x}{\sqrt{x^T \Sigma x}}\right) \geq p$$

Φ: Cumulative distribution function of standard normal distribution

G linear / Normal Distribution

Chance-Constrained Stochastic Programming Problem

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \quad \mathbb{P}\{\chi^T x \leq b\} \geq p$$

$\chi \in \mathbb{R}^n$: random vector with normally distr. entries
$\chi \sim \mathcal{N}(\mu, \Sigma)$
Σ: Covariance Matrix of χ

Deterministically reformulated problem

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t. } \Phi^{-1}(p) \cdot \sqrt{x^T \Sigma x} + \mu^T x \leq b$$

Φ: Cumulative distribution function of standard normal distribution

G linear / Normal Distribution / Independence

Deterministically reformulated problem

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \;\; \Phi^{-1}(p) \cdot \sqrt{x^T \Sigma x} \leq -\mu^T x + b$$

Second-Order Cone constraint

$\delta := \Phi^{-1}(p) > 0 \Leftrightarrow p > 0.5$

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \;\; \|x^T \Sigma^{\frac{1}{2}}\| \leq -\frac{1}{\delta}\mu^T x + \frac{b}{\delta}$$

G linear / Normal Distribution / **Independence**

Second-Order Cone constraint

$\delta := \Phi^{-1}(p) > 0 \Leftrightarrow p > 0.5$

$$\min_{x \in X} \quad f(x)$$

$$\text{s.t.} \quad \|x^T \Sigma^{\frac{1}{2}}\| \leq -\frac{1}{\delta}\mu^T x + \frac{b}{\delta}$$

Properties

- $X = \mathbb{R}^n \Rightarrow$ Feasible set is convex
- $X = \mathbb{R}^n$ & f convex $\Rightarrow$ Convex deterministic optimization problem
- If $X = \mathbb{R}^n$ & f linear $\Rightarrow$ Second-Order Cone Problem
  Solvable in polynomial time using Interior point method

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G linear / Poisson Distribution / Independence

Chance-Constrained Stochastic Programming Problem

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \quad \mathbb{P}\{\chi^T x \leq b\} \geq p$$

$\chi \in \mathbb{R}^n$: random vector with independent Poisson distr. entries
μ: Vector of means

Idea

- Find λ s.t. for $X \sim Pois(\lambda)$: $\quad \mathbb{P}\{X \leq b\} = p$
- Solve:

$$\min_{x \in X} \quad f(x)$$
$$\text{s.t.} \quad \mu^T x \leq \lambda$$

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Definition (Concave Function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **concave** if for all $x, y \in dom(f)$ and all $t \in [0, 1]$ it is

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$$

Definition (Log-Concave Function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **logarithmic concave (log-concave)** if $\log(f)$ is a concave function.

Property

A function $f : \mathbb{R}^n \to \mathbb{R}$ is log-concave iff for all $x, y \in dom(f)$ and all $\theta \in [0, 1]$ it is

$$f(\theta x + (1-\theta)y) \geq f(x)^{\theta} f(y)^{(1-\theta)}$$

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Definition (Log-Concave Function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called log-concave if $\log(f)$ is a concave function.

Definition (Log-Concave Probability Distribution)

A continuous probability distribution is called a **log-concave probability distribution** if the corresponding density function is log-concave.

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Theorem (Prékopa '72)

Let $g_i(x, y)$ $(i = 1, \ldots, m)$ be **concave functions** *on $\mathbb{R}^n \times \mathbb{R}^s$ (where x is an n-dimensional and y an s-dimensional vector). Let further χ be an s-dimensional random vector with* **logarithmic concave probability distribution***. Then, the left hand side x-function of the joint chance-constraint*

$$\mathbb{P}\{g_i(x, \chi) \geq 0, i = 1, \ldots, r\} \geq p \tag{1}$$

is **logarithmic concave** *in the entire space $\mathbb{R}^n$.*

Corollary (Prékopa '72)

Let $g_i(x, y)$ $(i = 1, \dots, m)$ be **quasi-concave functions** *on $\mathbb{R}^n \times \mathbb{R}^s$ (where x is an n-dimensional and y an s-dimensional vector). Let further χ be an s-dimensional random vector with* **logarithmic concave probability distribution***. Then the joint chance-constraint*

$$\mathbb{P}\{g_i(x, \chi) \geq 0, i = 1, \dots, r\} \geq p \tag{2}$$

defines a **convex set***.*

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Definition (Quasi-Concave Function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **quasi-concave** if for all $x, y \in dom(f)$ and all $t \in [0, 1]$ it is

$$f(tx + (1-t)y) \geq min(f(x), f(y))$$

Generalization by Tamm ('76/'77)

Prékopa's results stay valid if the g_i's are only quasi-concave!

Corollary (Prékopa '72)

Let $g_i(x, y)$ $(i = 1, \ldots, m)$ be **quasi-concave functions** *on $\mathbb{R}^n \times \mathbb{R}^s$ (where x is an n-dimensional and y an s-dimensional vector). Let further χ be an s-dimensional random vector with* **logarithmic concave probability distribution**. *Then the joint chance-constraint*

$$\mathbb{P}\{g_i(x, \chi) \geq 0, i = 1, \ldots, r\} \geq p \tag{2}$$

defines a **convex set**.

Examples of log-concave probability distributions

- Uniform distribution
- Normal distributions
- Exponential distribution
- Laplace distribution
- ...

QUESTIONS?

What about ~~next week~~ in 2 weeks?