Stochastic Optimization
IDA PhD course 2011ht

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2. Lecture: Uncertainties in objective
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1. Randomness occurs in the objective function
   - Expected value objective function
   - Probability of shortfall
   - Minimize Variance
   - Value at risk

2. One more SP example
   - Machine Scheduling

3. A bit of History

Outline

1. Randomness occurs in the objective function
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Deterministic Opt. Model → Stochastic Programming Model

$$\begin{aligned} \min_{x \in X} \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned} \qquad \rightarrow \qquad \begin{aligned} \min_{x \in X} \quad & f(x, \chi) \\ \text{s.t.} \quad & g(x, \chi) \leq 0 \end{aligned}$$

$\chi \in \Omega \subseteq \mathbb{R}^s$: vector with random variable as entries

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Outline

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Minimize an expected value function

$$\begin{aligned} \min_{x \in X} \quad & \mathbb{E}\left[f(x,\chi)\right] \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

Examples

- Expected cost / Expected gain
- Expected machine working time
- Expected transportation time
- Expected customer waiting times
- Expected damage on target

Advantages

- Good result "on average"
- Objective function can often be reformulated deterministically
- Convex objective if $f(\cdot, \chi)$ is convex (for all possible χ)
- Lower bound using Jensen's inequality:

Theorem (Jensen, 1906)

Let f be a convex function and X a random variable. Then

$$\mathbb{E}\left[f(X)\right] \geq f(\mathbb{E}\left[X\right])$$

Disadvantages

- We might encounter very "bad cases" ("Risk")
- Expectation can only be computed as multidimensional integral

Linear Programming Problem

Stochastic Programming Problem

$$\min_{\substack{x \in \mathbb{R}^n \\ x \geq 0}} \quad \mathbb{E}\left[c(\chi)^T x\right]$$

$$\text{s.t.} \quad Ax \leq b$$

$\chi \in \mathbb{R}^s$: random vector

Deterministically Reformulated Programming Problem

$$\min_{\substack{x \in \mathbb{R}^n \\ x \geq 0}} \quad \mu^T x$$

$$\text{s.t.} \quad Ax \leq b$$

$\mu \in \mathbb{R}^n$: (deterministic) vector of means

Discrete Finite Distribution

Stochastic Programming Problem

$$\min_{x \in X} \quad \mathbb{E}\left[f(x,\chi)\right]$$
$$\text{s.t.} \quad g(x) \leq 0$$

$\chi \in \mathbb{R}^s$: random vector

Deterministically Reformulated Programming Problem

$$\min_{x \in X} \quad \sum_{k=1}^{K} p^k f(x,\chi^k)$$
$$\text{s.t.} \quad g(x) \leq 0$$

$\chi^1, \ldots, \chi^K \in \mathbb{R}^s$: scenarios
$\mathbb{P}\{\chi = \chi^k\} := p^k$: probabilities

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General problem with discrete finite distributions

Exponential number of scenarios

Assume:

- Discretely distributed random variables
- Independently distributed random variables
- (Linear) Dependence: # dec. variables $\leftrightarrow$ # rand. variables

$\Longrightarrow$ Exponential number of scenarios

General problem with discrete finite distributions

Exponential number of scenarios

Assume:

- Discretely distributed random variables
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- (Linear) Dependency: # dec. variables $\leftrightarrow$ # rand. variables

$\Longrightarrow$ Exponential number of scenarios

Example

- n decision variables
- n random variables
- 2 possible outcomes for each random variable (e.g. Bernoulli distribution)

Independent random variables $\Rightarrow 2^n$ scenarios

Discrete Finite Distribution

Stochastic Programming Problem

$$\begin{aligned} \min_{x \in X} \quad & \mathbb{E}\left[f(x, \chi)\right] \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

$\chi \in \mathbb{R}^s$: random vector

Deterministically Reformulated Programming Problem

$$\begin{aligned} \min_{x \in X} \quad & \sum_{k=1}^{K} p^k f(x, \chi^k) \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

$\chi^1, \ldots, \chi^K \in \mathbb{R}^s$: scenarios
$\mathbb{P}\{\chi = \chi^k\} := p^k$: probabilities

Outline

Minimize probability of shortfall

$$\begin{aligned} \min_{x \in X} \quad & \mathbb{P}\{f(x,\chi) > T\} \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

Examples

- Investment strategies
- Project cost management ($T = 0$)

Probability of "Target" achievement

Advantages

- If probability of shortfall too high actions can be taken.

Disadvantages

- We might still encounter very "bad cases"
- No influence on average cost

Discrete Finite Distribution

Stochastic Programming Problem

$$\min_{x \in X} \quad \mathbb{P}\{f(x,\chi) > T\}$$
$$\text{s.t.} \quad g(x) \leq 0$$

$\chi \in \mathbb{R}^s$: random vector

$\chi^1, \ldots, \chi^K \in \mathbb{R}^s$: scenarios
$\mathbb{P}\{\chi = \chi^k\} := p^k$: probabilities

Reformulate Problem Deterministically

Basic idea:

- "Choose" scenarios with shortfall
- Probability that one of these arises minimized

Realization:

- Introduce one binary decision variable z^k per scenario
- $z^k = 1$: shortfall in scenario k

Discrete Finite Distribution II

Reformulate Problem Deterministically

Basic idea:

- "Choose" scenarios with shortfall
- Probability that one of these arises minimized

Realization:

- Introduce one binary decision variable z^k per scenario
- $z^k = 1$: shortfall in scenario k

Deterministically reformulated problem

$$\begin{aligned} \min \quad & \sum_{k=1}^{K} p^k z^k \\ \text{s.t.} \quad & g(x) \leq 0 \\ & f(x, \chi^k) \leq T + M z^k \quad \forall k = 1, \ldots, K \\ & x \in X, \quad z^k \in \{0, 1\} \quad \forall k = 1, \ldots, K \end{aligned}$$

M: some "big" constant

Discrete Finite Distribution III

Deterministically reformulated problem

$$\begin{aligned} \min \quad & \sum_{k=1}^{K} p^k z^k \\ \text{s.t.} \quad & g(x) \leq 0 \\ & f(x, \chi^k) \leq T + M z^k \quad \forall k = 1, \ldots, K \\ & x \in X, \quad z^k \in \{0, 1\} \quad \forall k = 1, \ldots, K \end{aligned}$$

M: some "big" constant

Problems

- Numerical instability due to big M possible
- K additional constraints
- K additional binary decision variables
- Deterministic reformulation hard

f linear / Normal Distribution

Stochastic Programming Problem

$$\begin{aligned} \min_{x \in X} \quad & \mathbb{P}\{\chi^T x > T\} \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

$\chi \in \mathbb{R}^n$: random vector with normally distr. entries
$\chi \sim \mathcal{N}(\mu, \Sigma)$
Σ: Covariance Matrix of χ

Deterministically reformulated problem

$$\begin{aligned} \max_{x \in X} \quad & \frac{T - \mu^T x}{\sqrt{x^T \Sigma x}} \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

$x^* \neq 0$

Outline

Minimize variance ?

$$\begin{aligned} \min_{x \in X} \quad & Var\left[f(x, \chi)\right] \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

Advantages:

- Outcome more concentrated around mean
- Possibility to reduce risk

Disadvantages:

- Makes not much sense without benchmark for expected costs

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Simple Mean-Variance Models

Minimize convex combination of variance and expectation

$$\begin{aligned} \min_{x \in X} \quad & \lambda Var\left[f(x,\chi)\right] + (1-\lambda)\mathbb{E}\left[f(x,\chi)\right] \qquad (\,\lambda \in (0,1)\,) \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

Minimize weighted product of variance and expectation

$$\begin{aligned} \min_{x \in X} \quad & Var\left[f(x,\chi)\right]^{\lambda} \cdot \mathbb{E}\left[f(x,\chi)\right] \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

Minimize variance with expectation threshold

$$\begin{aligned} \min_{x \in X} \quad & Var\left[f(x,\chi)\right] \\ \text{s.t.} \quad & g(x) \leq 0 \\ & \mathbb{E}\left[f(x,\chi)\right] \leq T \end{aligned}$$

Problems when variance in objective

- Loss of linearity
- Loss of convexity
- Hardness of problem (e.g. quadratic objective)
- Compute variance / Evaluate objective function

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Question

What is the probability that my total loss during a fixed time interval does not exceed a certain limit?

Examples

- What is the probability that my stock portfolio will fall in value by more than $ 100 million in one week?
- If I invest $ 1 million today, how much can I loose till tomorrow?

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Definition (Value-at-Risk)

X: random variable describing the loss over time horizon T
Φ_X: Cumulative distribution function of X

Value at risk over time horizon T at confidence level α:

$$VAR_\alpha(X) = \inf\{c|\Phi_X(c) \geq \alpha\}$$

Interpretation (Philippe Jorion)

"Value at Risk measures the worst expected loss over a given horizon under normal market conditions at a given level of confidence."

Value at Risk in Stochastic Programming

- Risk measure
- Objective: Minimize value at risk

Critics

- Lack of subadditivity
- Lack of convexity
- Difficult to compute from scenarios

Alternatives

- Conditional value at risk
- Tail value at risk
- ...

Outline

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Outline

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Deterministic Problem

(Possible) Parameters

- # of (different) machines / parts
- Processing times
- # of jobs to be completed
- # of employees available
- Due dates
- Precedence relations

(Possible) Objectives

- Minimize total completition time
- Maximize # of completed jobs
- Minimize maximum/sum of tardyness
- Minimize idle times

Stochastic Problem

(Possible) Uncertain Parameters

- # of (different) available machines ← break downs
- # of (different) parts ← costumization
- Processing times ← manual operations
- # of jobs to be completed ← demand
- # of employees available ← sickness, vacations
- Due dates ← uncertainty in processing times
- ~~Precedence relations~~

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Stochastic Problem

(Possible) Objective

- minimize expected... total processing time
- Given # of jobbs, maximize probability that... processing "in time"

Stochastic Settings

- Single stage decision
- Multi-Stage decision $\leftarrow$ Discretization of processing time
- Online Programming $\leftarrow$ New information arrives over time

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The beginning

George Dantzig
Linear programming under uncertainty. (1955)
Management Science 1:197–206

- Two-Stage and Simple recourse problems
- Finite number of scenarios
- Deterministic Reformualtion
- No use of special structure

Richard Van Slyke and Roger J-B. Wets
L-shaped linear programs with applications to optimal control and stochastic programming. (1969)
MSIAM Journal on Applied Mathematics 17(4):638–663, 1969

- Solution method that makes use of special problem structures
- Reduced computing time

András Prékopa
On probabilistic constrained programming. (1970)
Proceedings of the Princeton Symposium on Mathematical Programming 113–1383

András Prékopa
A class of stochastic programming decision problems. (1972)
Mathematische Operationsforschung und Statistik 3(5):349–354

- Main contributions to understanding of chance-constraint programming
- Convex cases
- Joint constraints

Maarten H. van der Vlerk
Stochastic Programming with Integer Recourse. (1995)
PhD thesis, University of Groningen, The Netherlands

- Main contributions to understanding of Integer Programming with Recourse
- with Leen Stougie, Rüdiger Schultz

Alexander Shapiro and Tito Homem-de-Mello
A simulation-based approach to two-stage stochastic programming with recourse. (1998)
Mathematical Programming 81(3):301-325

- Stochastic Programming via Monte Carlo Sampling: Sample Average Approach
- Much work on convergence properties
- Realization: Often good approximations possible with "relatively" few samples

Next lecture

- Chance-Constrained Programming and related problems
- (Simple Recourse Problems)

QUESTIONS?

What about ~~next week~~ in 2 weeks?