Stochastic Optimization
IDA PhD course 2011ht

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1. Lecture: Introduction
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Outline

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Examiner/"Owner": Peter Jonsson
Lecturer: Stefanie Kosuch
Credits: 5hp

Examination

Oral Examination:

- Presentation to be prepared
- Article Review, Practical Assignment, Theoretical Assignment...
- 10-15min: Presentation
- 10-15min: Questions

Schedule

2011/09/29: 1st lecture
2011/10/06: 2nd lecture
2011/10/13: "vacation"
2011/10/20: 3rd lecture
2011/10/27: "Subject Research"
2011/11/03: 4th lecture
2011/11/10: 5th lecture - Examination Subjects
2011/11/17: 6th lecture
2011/11/24: 7th lecture - General Questions on Examination
2011/12/01: 8th lecture
2011/12/08: 9th lecture
2011/12/15: 10th lecture
2012/12/19: First preparation week
2012/01/02: Second preparation week
2012/01/09: Examination week

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Course Material

- Slides (same day or day $+1$)
- Alexander Shapiro, Darinka Dentcheva, Andrzej Ruszczyński
  Lectures on Stochastic Programming (2009)
  pdf available online
- Peter Kall, Stein W. Wallace
  Stochastic Programming (1994)
  pdf available online
- András Prékopa
  Stochastic Programming (1995)
  Springer.
- Evtl. Additional articles ("further reading")

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Definition

Stochastic Programming: Study and resolution of Optimization problems that involve uncertainties.
Stochastic Optimization: Concerns Solution methods that "involve" random variables.
Stochastic Combinatorial Optimization: Study and resolution of Combinatorial Optimization problems that involve uncertainties.

Aims

- Examples of Optimization Problems that involve uncertainty
- Modeling Optimization Problems that involve uncertainty
- Special Structure of Stochastic Programming (SP) Problems
- Hardness of Stochastic Programming Problems
- Solving Stochastic Programming Problems

Course Contents

1. Introduction:
   - Why Stochastic Programming?
   - History
   - Small Examples of SP problems
   - General Structure and basic concepts
2. Uncertainty in Objective Function - Models, Structure and Hardness
3. Uncertainty in Constraint - Models, Structure and Hardness
   a) Chance-Constrained Programming
   b) Simple Recourse Programming
   c) Two-Stage Programming
   d) Multi-Stage Programming
4. Algorithms for Stochastic Programming Problems:
   a) Stochastic Gradient Methods
   b) Decomposition Methods
   c) Stochastic Decomposition
5. Sample Average Approach
6. More "realistic" Examples

Optional Contents

- Robust/Online Optimization
- Approximation Algorithms
- Heuristics for Stochastic Programming Problems

Structure

Notations: "On the fly"
Repetitions: Random Distributions & Complexity
(*PLEASE REVISE INDIVIDUALLY!*)
Proofs: Basic Ideas / cruxes
Questions: Immediately

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What is the interest of Stochastic Programming?

"Real world problems" often subject to uncertainties

- Not all parameters known when decision has to be made: market fluctuations, available capacity...
- Own decision depends on future decision of other parties: competition, clients, government...
- Setting of problem might change: weather, location...

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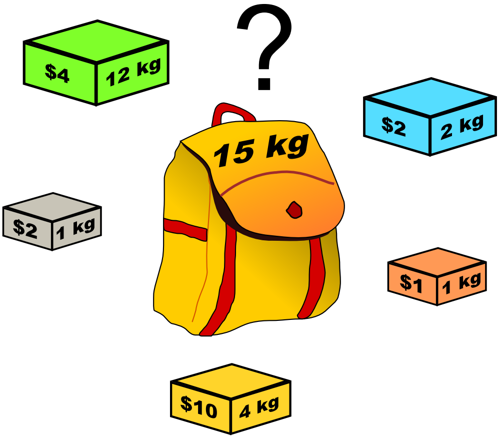
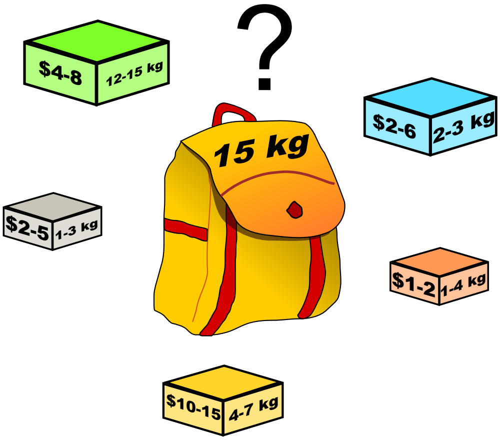
Deterministic Knapsack problem



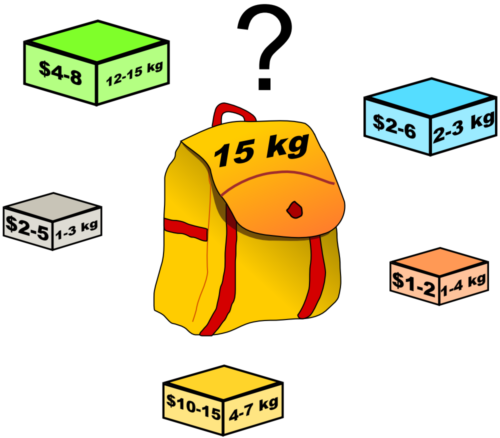
Stochastic Knapsack problem



Possible ways to handle capacity constraint

- knapsack constraint violated $\Rightarrow$ penalty
- probability of capacity violation restricted
- decision can be corrected later (add. costs/reduced rewards)

Stochastic Knapsack problem



Possible ways to handle capacity constraint

- knapsack constraint violated $\Rightarrow$ penalty
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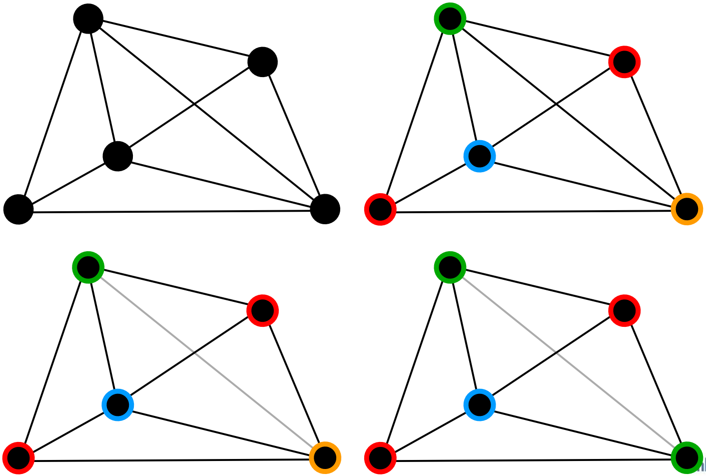
Stochastic Graph Coloring



Changing settings

- set of edges random
- set of vertices random

Changing parameters

- allowed number of colors random
- "cost" of colors random

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Deterministic Opt. Model → Stochastic Progr. Model

$$\begin{array}{lll} \max\limits_{x \in X} & f(x) & \\ \text{s.t.} & g(x) \leq 0 & \end{array} \quad \rightarrow \quad \begin{array}{ll} \min\limits_{x \in X} & f(x, \chi) \\ \text{s.t.} & g(x, \chi) \leq 0 \end{array}$$

$\chi \in \Omega \subseteq \mathbb{R}^s$: vector with random entries

Difficulties in SP - Modeling

- Information on uncertainty
  - Statistics
  - Simulation
- What can we assume?
  - Full knowledge of random variables
  - Approximation of (continuous) distribution
  - Work with finite sample
- Which Model to choose?
  - Uncertainty in Objective or Constraints
  - What means "feasible"?
  - Can decision be made in stages?

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Difficulties in SP - Resolution

- Structural Problem
  - non-convexity
  - non-continuity
  - non-analytic expressions
- Problem Size
- Deterministic or Random Method?

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Greatest Challenges in Stochastic Programming

- Modeling Real World problems with uncertainties as SP problems
- Solve "realistic" sized problems in reasonable time
- Find more adapted solution techniques (for general distributions)

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I: Approximate Distribution

1. Observations of random parameters
2. "Extract" representative sample
3. Approximate distribution:
   a) Either: Work with discrete sample
   b) Or: Approximate by continuous distribution

Main Problem

Inexact Approximation

II: Work with sampling

1. Keep distribution unknown in model
2. "Online" samples in solution process
3. $\rightarrow$ Black Box Model

Main Problem

Evaluation of objective/constraint function difficult

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Notations

- $\mathbb{P}\{A\}$: Probability that event A arises
- $\mu := \mathbb{E}[X]$: expectation of random variable X
- $\sigma := \sqrt{V(X)}$: standard deviation of random variable X
- $\varphi_X(x)$:
  Density function of *continuous* random variable X
  Probability mass function of *discrete* random variable X
- $\Phi_X(c) = \mathbb{P}\{X \leq c\}$: Cumulative distribution function of random variable X

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Random vectors

- $\chi \in \mathbb{R}^n$: Random vector
- $\chi = (\chi_1, \ldots, \chi_n)$
- χ_i: Random variable
- Dependent or independent?
- $A(\chi) \in \mathbb{R}^{n_1 \times m_1}$: Matrix with random entries

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Important facts on continuous distributions

- Cumulative distribution continuous
- Density function continuous on certain interval(s)
- $\forall a \in (-\infty, \infty) : \mathbb{P}\{X = a\} = 0$
- $$\mathbb{E}[X] = \int_{-\infty}^{\infty} y \cdot \varphi_X(y) dy$$

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Normal distribution

Density function:

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Cumulative distribution function:

$$\Phi_X(c) = ?$$

Examples

- "Natural" measures (body height, shoe size, tree heights...)
- Economic phenomena (e.g. income distribution)
- Data Measurements

Uniform distribution

Density function:

$$\varphi_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Cumulative distribution function:

$$\Phi_X(c) = \begin{cases} 0 & \text{if } c < a \\ \dfrac{c-a}{b-a} & \text{if } c \in [a, b] \\ 1 & \text{if } c > b \end{cases}$$

Examples

- Waiting time for bus
- Leak between two access points to pipeline (appr.)

Exponential distribution

Density function:

$$\varphi_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Cumulative distribution function:

$$\Phi_X(c) = \begin{cases} 1 - e^{-\lambda c} & \text{if } c \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Examples

- "Lifetime" (light bulbs, batteries, electronic components...)
- Time till next earthquake
- Amount of change in your pocket
- Phone calls (length, time between two calls...)

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Some facts on discrete distributions

- Generally concern counts ("Number of...")
- $\exists$ countable set $S = S(X)$ such that
  - $\mathbb{P}\{X = a\} > 0 \Leftrightarrow a \in S$
  - $\varphi_X(x) = \mathbb{P}\{X = x\}$
  - $\mathbb{E}[X] = \sum_{y \in S} y \, \mathbb{P}\{X = y\}$
- Cumulative distribution function increases only by jump discontinuities

Discrete finite distribution

Probability mass function:
$\exists\ S$ with $|S| < \infty$ such that

$$\mathbb{P}\{X = x\} > 0 \Leftrightarrow x \in S$$

Cumulative distribution function:

$$\Phi_X(c) = \sum_{\substack{x \in S \\ x \leq c}} \mathbb{P}\{X = x\}$$

Notations

- $K = |S|$ possible realizations or "scenarios"
- $X^1, \ldots, X^K$

Discrete finite distribution

Probability mass function:
$\exists\ S$ with $|S| < \infty$ such that

$$\mathbb{P}\{X = x\} > 0 \Leftrightarrow x \in S$$

Cumulative distribution function:

$$\Phi_X(c) = \sum_{\substack{x \in S \\ x \leq c}} \mathbb{P}\{X = x\}$$

Examples

- Discrete random variables with bounds (e.g. utilized capacity)
- Approximation of (continuous) distribution by finite sample

Discrete uniform distribution

Probability mass function:
$\exists\ S$ with $|S| < \infty$ such that

$$\mathbb{P}\{X = x\} = 1/|S| \Leftrightarrow x \in S$$

Cumulative distribution function:

$$\Phi_X(c) = \sum_{\substack{x \in S \\ x \leq c}} 1/|S|$$

Examples

- Throwing a die
- Lottery

Bernoulli distribution

Probability mass function:

$$\mathbb{P}\{X = 1\} = p \text{ and } \mathbb{P}\{X = 0\} = 1 - p$$

Cumulative distribution function:

$$\Phi_X(c) = \begin{cases} 0 \text{ if } c \in (-\infty, 0) \\ 1 - p \text{ if } c \in [0, 1) \\ 1 \text{ if } c \in [1, \infty) \end{cases}$$

Examples

- Success/Failure
- "Yes/No" Events

Poisson distribution

Probability mass function:

$$\varphi_X(k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad (k \in \mathbb{Z})$$

Cumulative distribution function:

$$\Phi_X(c) = \sum_{i=0}^{c} \frac{\lambda^i e^{-\lambda}}{i!}$$

Appropriate model for counts

- # phone calls
- # type errors on a page
- # white blood cells found in a cubic centimeter of blood
- Large $\lambda \rightarrow$ normal distribution

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Decision Problems

- **P:** Polynomial time solvable
- **NP:** Verification of "yes"-instance in polynomial time
- **NP-hard:** "at least as hard as the hardest problems in NP"
- **NP-complete:** NP-hard problems in NP

?

$P \neq NP$

Counting Problems

- $\sharp P$**:** Counting problems associated with problems in NP
- $\sharp P$**-complete:**
  In $\sharp P$ and every problem in $\sharp P$ can be reduced to it

$\sharp P$-complete problems

- "How many graph colorings using k colors are there for a particular graph G?"
- "How many perfect matchings are there for a given bipartite graph?"

$\sharp P$-complete problem solvable in pol. time $\Rightarrow P = NP$

Next lecture

- A little bit of history.
- (Some more small examples.)
- Uncertainty in the objective function.

QUESTIONS?

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