Stochastic Optimization
IDA PhD course 2011ht

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10. Lecture: Computational examples
09. February 2012

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Outline

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Is-Situation

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- Use average/expecteted values

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- No adaptations to current situation

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- Average gain $\neq$ computed expected value

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- Solution not robust

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Disadvantages

- Average gain $\neq$ computed expected value
- No flexibility
- Solution not robust
- Might entail infeasible solutions

Solutions

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- Online Optimization

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- Robust Optimization

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- Stochastic Programming

Solutions

- Online Optimization ?
- Robust Optimization
- Stochastic Programming
- ...

Typical application fields

- Capacity planning
- Energy sector
- Finance
- Forestry
- Military
- Production / Supply chain
- Scheduling
- Transportation (of humans, goods,...)
- Water management
- ...

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Example: Continuous Knapsack with uncertain capacity

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- $a = (5, 10, 20)$
- $p_1 = \mathbb{P}\{\chi = \chi_1\} = \mathbb{P}\{\chi = 3\} = 0.2$

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- $p_2 = \mathbb{P}\{\chi = \chi_2\} = \mathbb{P}\{\chi = 12\} = 0.3$

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- $p_3 = \mathbb{P}\{\chi = \chi_3\} = \mathbb{P}\{\chi = 25\} = 0.5$

Scenario dependent solutions

Scenario dependent solutions

$$\begin{aligned} \max_{x\in[0,1]^3} \quad & c^T x(\chi) \\ \text{s.t.} \quad & a^T x(\chi) \le \chi \end{aligned}$$

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Scenario dependent solutions

$$\max_{x\in[0,1]^3} \quad c^T x(\chi)$$
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- $x(3) = (0.6; 0; 0)^T$; $z(3) = 6$

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- $x(25) = (1; 1; 0.5)^T$; $z(25) = 35$

Scenario dependent solutions

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- $x(12) = (1; 0.7; 0)^T$; $z(12) = 20.5$
- $x(25) = (1; 1; 0.5)^T$; $z(25) = 35$
- $z = \mathbb{E}[z(\chi)] = \sum_{i=1}^{3} p_i z(\chi) = 24.85$

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Average value approach

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$$\max_{x \in [0,1]^3} \quad c^T x$$
$$\text{s.t.} \quad a^T x \leq \mathbb{E}[\chi]$$

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$$\max_{x \in [0,1]^3} \quad c^T x$$
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Average value approach

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- $x = (1; 1; 0.085)^T$
- $z = 26.7$

Worst case approach

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$$\max_{x \in [0,1]^3} \quad c^T x$$
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Chance Constrained Approach

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- $z = 20.5$

Two-Stage Approach

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$$\begin{aligned} \max_{x \in [0,1]^3} \quad & c^T x - \sum_{i=1}^{3} p_i q(\chi_i) y(\chi_i) \\ \text{s.t.} \quad & a^T x - y(\chi_i) \leq \chi_i \quad \forall i \end{aligned}$$

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- $z = 17.5$

Simple Recourse Approach

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$$\max_{x \in [0,1]^3} \quad c^T x - \mathbb{E}[q(\chi)[a^T x - \chi]^+]$$

Simple Recourse Approach

$$\max_{x \in [0,1]^3} \quad c^T x - \mathbb{E}[q(\chi)[a^T x - \chi]^+]$$

- $z = 17.5$

Outline

Reference

J. Balasubramanian and I. E. Grossmann
Approximation to Multistage Stochastic Optimization in Multiperiod Batch Plant Scheduling under Demand Uncertainty. (2003)
http://egon.cheme.cmu.edu/Papers/BalasubMultistage.pdf

Multiproduct, multiperiod Batch production

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- Different products

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- Different stages

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- Different costs (materials, stockage/holding, under-production...)

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- Demand
- Processing times

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Uncertainties

- Demand
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Online Optimization?

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- Long processing times
- Need to make predictions
- Holding/storage costs $\leftrightarrow$ idle costs
- Difficult mathematical problem $\leftrightarrow$ reaction times

Example Problem

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- single-stage single-unit bath plant

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- 2 products

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- 3 processing modes (batch sizes)

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Example Problem

- single-stage single-unit bath plant
- 2 products
- 3 processing modes (batch sizes)
- Uncertain demand
- 2 time periods (same demand distribution)

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Product	Batch size (tons)	Proc. time	REV ($/tons)	XC ($/tons)	LC ($/tons)
A	0-5 5-10 10-25	2 4 6	100	10	20
B	0-5 5-10 10-25	3 5 7	250	20	50

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Event	Probability	Demands (tons)
1	0.25	A:10, B:0
2	0.75	A:20, B:5

Deterministic approach

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- Work with expected demands over entire horizon

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- Model predicts: $5375

Deterministic approach

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- Model predicts: $5375
- Actual expected profit: $4559

Two-Stage approach

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- First-stage: production scheduling over both time periods

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- Second-stage: Amount to be sold / Unsatisfied amount

Two-Stage approach

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- Second-stage: Amount to be sold / Unsatisfied amount
- Expected profit: \$5275 (+15%)

Three-Stage approach

Three-Stage approach

- First-stage: production scheduling over 1. period

Three-Stage approach

- First-stage: production scheduling over 1. period
- Second-stage: production scheduling over 2. period

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Three-Stage approach

- First-stage: production scheduling over 1. period
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Three-Stage approach

- First-stage: production scheduling over 1. period
- Second-stage: production scheduling over 2. period
- Third-stage: Amount to be sold / Unsatisfied amount
- Expected profit: \$5325 (+17%)

Idea

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- Approximate solution

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- Shrinking time horizon strategy

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- Basic structure

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  - Solve two-stage problem over remaining time horizon (given past demands)

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- Problem solved in advance! ($\neq$ online optimization)

Idea

- Approximate solution
- Shrinking time horizon strategy
- Basic structure
  - Solve two-stage problem over remaining time horizon (given past demands)
  - Implement solution *only* in next time period

!

- Problem solved in advance! ($\neq$ online optimization)
- Two-Stage problem for all nodes in scenario tree

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1. Numerical Example

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- single product

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- single product
- 3 stages

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- single product
- 3 stages
- 3 processing modes each

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- single product
- 3 stages
- 3 processing modes each
- 5 time periods (different probability distributions)

Approach	Expected revenue (1000$)	+	CPU time (sec)
Det.	656.23	0%	3
2-stage	672.04	2%	20
SH	717.32	9%	360
6-stage	722.43	10%	>50,000

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2. Numerical Example

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- 4 products

2. Numerical Example

- 4 products
- 8 tasks

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- 4 products
- 8 tasks
- 6 processing units

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- 4 products
- 8 tasks
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2. Numerical Example

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- 8 tasks
- 6 processing units
- 3 processing modes for each combination
- 3 time periods (different probability distributions)

Approach	Expected revenue ($)	$+$	CPU time (sec)
Det.	59,573	0%	10
2-stage	62,945	6%	35
SH	75,452	27%	180
4-stage	75,851	27%	$>$100,000

Conclusion

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- Working with average values $\rightarrow$ false estimations
- Choosing two/multi-stage model: important increase in gain
- Approximations help to decrease CPU time
- Approximations still much better than det. model

QUESTIONS?

Tack sa mycket!