Stochastic Optimization
IDA PhD course 2011ht

Stefanie Kosuch
PostDok at TCSLab
www.kosuch.eu/stefanie/

1. Lecture: Introduction
29. September 2011

1 Administrative Details

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Outline

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Examiner/"Owner": Peter Jonsson

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Examiner/"Owner": Peter Jonsson
Lecturer: Stefanie Kosuch

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Lecturer: Stefanie Kosuch
Credits: 5hp

Examination

Oral Examination:

- Presentation to be prepared

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- Article Review, Practical Assignment, Theoretical Assignment...

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- 10-15min: Presentation

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Examination

Oral Examination:

- Presentation to be prepared
- Article Review, Practical Assignment, Theoretical Assignment...
- 10-15min: Presentation
- 10-15min: Questions

Schedule

2011/09/29: 1st lecture
2011/10/06: 2nd lecture
2011/10/13: "vacation"
2011/10/20: 3rd lecture
2011/10/27: "Subject Research"
2011/11/03: 4th lecture
2011/11/10: 5th lecture - Examination Subjects
2011/11/17: 6th lecture
2011/11/24: 7th lecture - General Questions on Examination
2011/12/01: 8th lecture
2011/12/08: 9th lecture
2011/12/15: 10th lecture
2012/12/19: First preparation week
2012/01/02: Second preparation week
2012/01/09: Examination week

Course Material

- Slides (same day or day $+1$)

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- Alexander Shapiro, Darinka Dentcheva, Andrzej Ruszczyński
  Lectures on Stochastic Programming (2009)
  pdf available online

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- András Prékopa
  Stochastic Programming (1995)
  Springer.
- Evtl. Additional articles ("further reading")

Outline

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Definition

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Stochastic Programming: Study and resolution of Optimization problems that involve uncertainties.

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Stochastic Optimization: Concerns Solution methods that "involve" random variables.

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Stochastic Programming: Study and resolution of Optimization problems that involve uncertainties.
Stochastic Optimization: Concerns Solution methods that "involve" random variables.
Stochastic Combinatorial Optimization: Study and resolution of Combinatorial Optimization problems that involve uncertainties.

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Aims

- Examples of Optimization Problems that involve uncertainty

Aims

- Examples of Optimization Problems that involve uncertainty
- Modeling Optimization Problems that involve uncertainty

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- Modeling Optimization Problems that involve uncertainty
- Special Structure of Stochastic Programming (SP) Problems

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- Special Structure of Stochastic Programming (SP) Problems
- Hardness of Stochastic Programming Problems

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Aims

- Examples of Optimization Problems that involve uncertainty
- Modeling Optimization Problems that involve uncertainty
- Special Structure of Stochastic Programming (SP) Problems
- Hardness of Stochastic Programming Problems
- Solving Stochastic Programming Problems

Course Contents

1. Introduction:

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1. Introduction:
   - Why Stochastic Programming?

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   - Small Examples of SP problems

Course Contents

1. Introduction:
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   - Small Examples of SP problems
   - General Structure and basic concepts

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2. Uncertainty in Objective Function - Models, Structure and Hardness

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   a) Chance-Constrained Programming

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4. Algorithms for Stochastic Programming Problems:

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   a) Stochastic Gradient Methods

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5. Sample Average Approach

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4. Algorithms for Stochastic Programming Problems:
   a) Stochastic Gradient Methods
   b) Decomposition Methods
   c) Stochastic Decomposition
5. Sample Average Approach
6. More "realistic" Examples

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Optional Contents

- Robust/Online Optimization

Optional Contents

- Robust/Online Optimization
- Approximation Algorithms

Optional Contents

- Robust/Online Optimization
- Approximation Algorithms
- Heuristics for Stochastic Programming Problems

Structure

Notations: "On the fly"

Structure

Notations: "On the fly"
Repetitions: Random Distributions & Complexity
(*PLEASE REVISE INDIVIDUALLY!*)

Structure

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Repetitions: Random Distributions & Complexity
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Proofs: Basic Ideas / cruxes

Structure

Notations: "On the fly"
Repetitions: Random Distributions & Complexity
(*PLEASE REVISE INDIVIDUALLY!*)
Proofs: Basic Ideas / cruxes
Questions: Immediately

Outline

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"Facts"

"Real world problems" often subject to uncertainties

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"Real world problems" often subject to uncertainties

- Not all parameters known when decision has to be made: market fluctuations, available capacity...

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- Own decision depends on future decision of other parties: competition, clients, government...

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"Facts"

"Real world problems" often subject to uncertainties

- Not all parameters known when decision has to be made: market fluctuations, available capacity...
- Own decision depends on future decision of other parties: competition, clients, government...
- Setting of problem might change: weather, location...

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What is the interest of Stochastic Programming?

"Real world problems" often subject to uncertainties

- Not all parameters known when decision has to be made: market fluctuations, available capacity...
- Own decision depends on future decision of other parties: competition, clients, government...
- Setting of problem might change: weather, location...

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Outline

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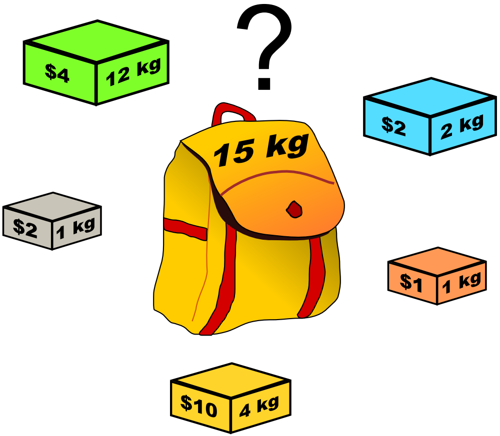
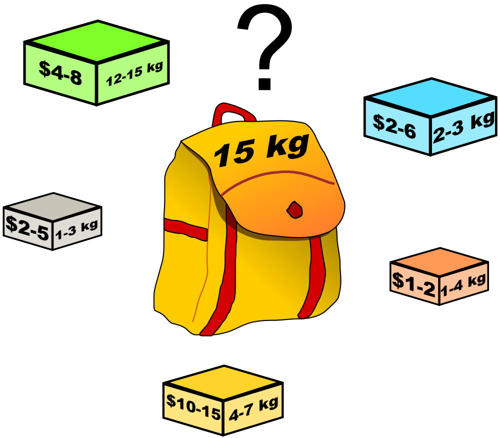
Deterministic Knapsack problem



Stochastic Knapsack problem



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Possible ways to handle capacity constraint

Possible ways to handle capacity constraint

- knapsack constraint violated $\Rightarrow$ penalty

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- knapsack constraint violated $\Rightarrow$ penalty
- probability of capacity violation restricted

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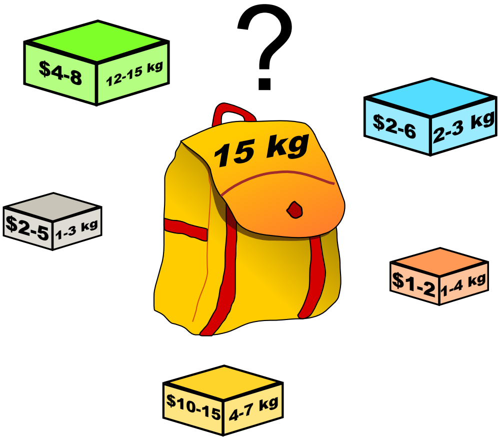

Possible ways to handle capacity constraint

- knapsack constraint violated $\Rightarrow$ penalty
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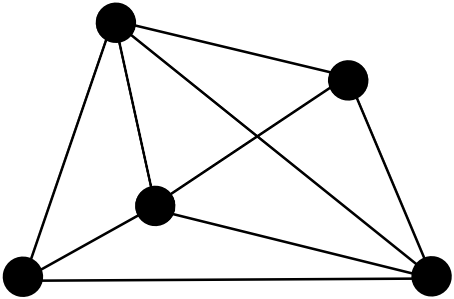
Possible ways to handle capacity constraint

- knapsack constraint violated $\Rightarrow$ penalty
- probability of capacity violation restricted
- decision can be corrected later (add. costs/reduced rewards)

Deterministic Graph Coloring



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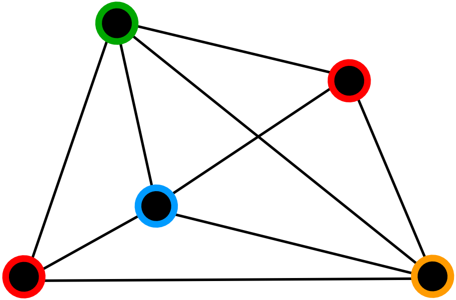

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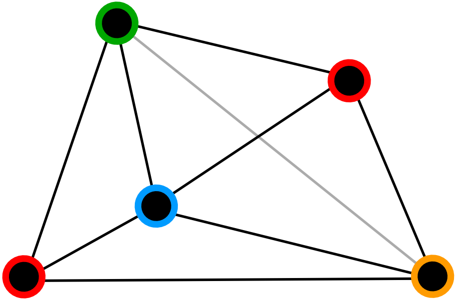

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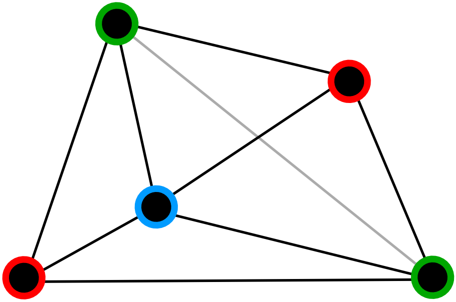

Changing settings

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- set of edges random

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Changing parameters

Changing settings

- set of edges random
- set of vertices random

Changing parameters

- allowed number of colors random

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Changing settings

- set of edges random
- set of vertices random

Changing parameters

- allowed number of colors random
- "cost" of colors random

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Outline

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Deterministic Opt. Model $\rightarrow$ Stochastic Progr. Model

$$\begin{aligned} \max_{x \in X} \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0 \end{aligned}$$

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Deterministic Opt. Model $\rightarrow$ Stochastic Progr. Model

$$
\begin{array}{ll} \max\limits_{x \in X} & f(x) \\ \text{s.t.} & g(x) \leq 0 \end{array} \quad \longrightarrow \quad \begin{array}{ll} \min\limits_{x \in X} & f(x, \chi) \\ \text{s.t.} & g(x, \chi) \leq 0 \end{array}
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$\chi \in \Omega \subseteq \mathbb{R}^s$: vector with random entries

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Difficulties in SP - Modeling

- Information on uncertainty

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  - Uncertainty in Objective or Constraints
  - What means "feasible"?
  - Can decision be made in stages?

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Difficulties in SP - Resolution

- Structural Problem

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Difficulties in SP - Resolution

- Structural Problem
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- Problem Size
- Deterministic or Random Method?

Greatest Challenges in Stochastic Programming

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- Modeling Real World problems with uncertainties as SP problems

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- Modeling Real World problems with uncertainties as SP problems
- Solve "realistic" sized problems in reasonable time

Greatest Challenges in Stochastic Programming

- Modeling Real World problems with uncertainties as SP problems
- Solve "realistic" sized problems in reasonable time
- Find more adapted solution techniques (for general distributions)

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Outline

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I: Approximate Distribution

1. Observations of random parameters

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Main Problem

Inexact Approximation

II: Work with sampling

1. Keep distribution unknown in model

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2. "Online" samples in solution process

II: Work with sampling

1. Keep distribution unknown in model
2. "Online" samples in solution process
3. $\rightarrow$ Black Box Model

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Main Problem

Evaluation of objective/constraint function difficult

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Notations

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- $\mathbb{P}\{A\}$: Probability that event A arises

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- $\mu := \mathbb{E}[X]$: expectation of random variable X

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- $\mathbb{P}\{A\}$: Probability that event A arises
- $\mu := \mathbb{E}[X]$: expectation of random variable X
- $\sigma := \sqrt{V(X)}$: standard deviation of random variable X

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Notations

- $\mathbb{P}\{A\}$: Probability that event A arises
- $\mu := \mathbb{E}[X]$: expectation of random variable X
- $\sigma := \sqrt{V(X)}$: standard deviation of random variable X
- $\varphi_X(x)$:
  Density function of *continuous* random variable X
  Probability mass function of *discrete* random variable X

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- $\sigma := \sqrt{V(X)}$: standard deviation of random variable X
- $\varphi_X(x)$:
  Density function of *continuous* random variable X
  Probability mass function of *discrete* random variable X
- $\Phi_X(c) = \mathbb{P}\{X \leq c\}$: Cumulative distribution function of random variable X

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Random vectors

Random vectors

- $\chi \in \mathbb{R}^n$: Random vector

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- $\chi \in \mathbb{R}^n$: Random vector
- $\chi = (\chi_1, \ldots, \chi_n)$

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- Dependent or independent?

Random vectors

- $\chi \in \mathbb{R}^n$: Random vector
- $\chi = (\chi_1, \ldots, \chi_n)$
- χ_i: Random variable
- Dependent or independent?
- $A(\chi) \in \mathbb{R}^{n_1 \times m_1}$: Matrix with random entries

Outline

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Important facts on continuous distributions

- Cumulative distribution continuous

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- Density function continuous on certain interval(s)

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Important facts on continuous distributions

- Cumulative distribution continuous
- Density function continuous on certain interval(s)
- $\forall a \in (-\infty, \infty) : \mathbb{P}\{X = a\} = 0$
- $$\mathbb{E}[X] = \int_{-\infty}^{\infty} y \cdot \varphi_X(y) dy$$

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Normal distribution

Normal distribution

Density function:

Normal distribution

Density function:

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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Cumulative distribution function:

Normal distribution

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Cumulative distribution function:

$$\Phi_X(c) = ?$$

Normal distribution

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$$\Phi_X(c) = ?$$

Examples

- "Natural" measures (body height, shoe size, tree heights...)

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Normal distribution

Density function:

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Cumulative distribution function:

$$\Phi_X(c) = ?$$

Examples

- "Natural" measures (body height, shoe size, tree heights...)
- Economic phenomena (e.g. income distribution)

Normal distribution

Density function:

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Cumulative distribution function:

$$\Phi_X(c) = ?$$

Examples

- "Natural" measures (body height, shoe size, tree heights...)
- Economic phenomena (e.g. income distribution)
- Data Measurements

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Uniform distribution

Uniform distribution

Density function:

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Density function:

$$\varphi_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

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Examples

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- Leak between two access points to pipeline (appr.)

Exponential distribution

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- Phone calls (length, time between two calls...)

Outline

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Some facts on discrete distributions

Some facts on discrete distributions

- Generally concern counts ("Number of...")

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- $\exists$ countable set $S = S(X)$ such that

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- Cumulative distribution function increases only by jump discontinuities

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Discrete finite distribution

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Probability mass function:

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- $X^1, \ldots, X^K$

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Examples

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Examples

- Discrete random variables with bounds (e.g. utilized capacity)
- Approximation of (continuous) distribution by finite sample

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Discrete uniform distribution

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Examples

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Examples

- Throwing a die
- Lottery

Bernoulli distribution

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Examples

- Success/Failure
- "Yes/No" Events

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Poisson distribution

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Appropriate model for counts

- # phone calls

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- Large $\lambda \to$ normal distribution

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Outline

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Decision Problems

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- **P:** Polynomial time solvable

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$P \subseteq NP$

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$\sharp P$-complete problem solvable in pol. time $\Rightarrow P = NP$

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Next lecture

- A little bit of history.

Next lecture

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- (Some more small examples.)

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Next lecture

- A little bit of history.
- (Some more small examples.)
- Uncertainty in the objective function.

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QUESTIONS?