Introduction to Stochastic Combinatorial Optimization

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Guest Lecture at the CUGS PhD course
"**Heuristic Algorithms for Combinatorial Optimization Problems**"

What is the interest of Stochastic Combinatorial Optimization?

Combinatorial "real world problems" often subject to uncertainties

- Not all parameters known when decision has to be made: market fluctuations, available capacity...
- Own decision depends on future decision of other parties: competition, clients, government...
- Setting of problem might change: weather, location...

Definition

Stochastic Combinatorial Optimization concerns the study and resolution of Combinatorial Optimization problems that involve uncertainties.

Objectives of this lecture

- Give you examples of SCO-problems.
- Give you an idea of how uncertainties can be modeled (most common models).
- Give you an idea of why Stochastic Optimization is hard.
- Give you an idea of how SCO-problems can be solved.
- Give you an idea of why metaheuristics are important tools in SCO.

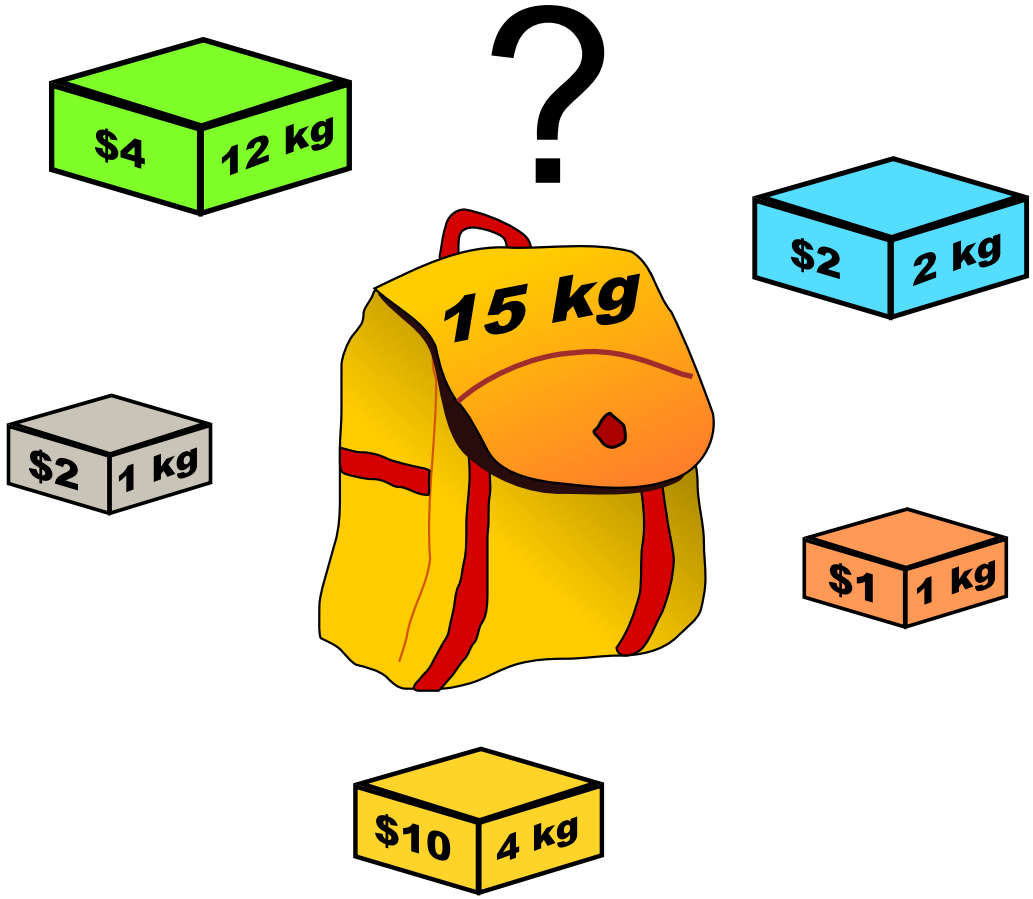
Outline

1 2 examples of SCO problems
2 Modeling Stochastic Combinatorial Optimization Problems
- "Where" does the randomness occur?
  - Randomness occurs in the objective function f.
  - Randomness occurs in the constraint function g.
- When are the actual parameters revealed?
  - Parameters are revealed after decision has been made.
  - Parameters are revealed before corrective decision is made.
  - Parameters are revealed in several stages.

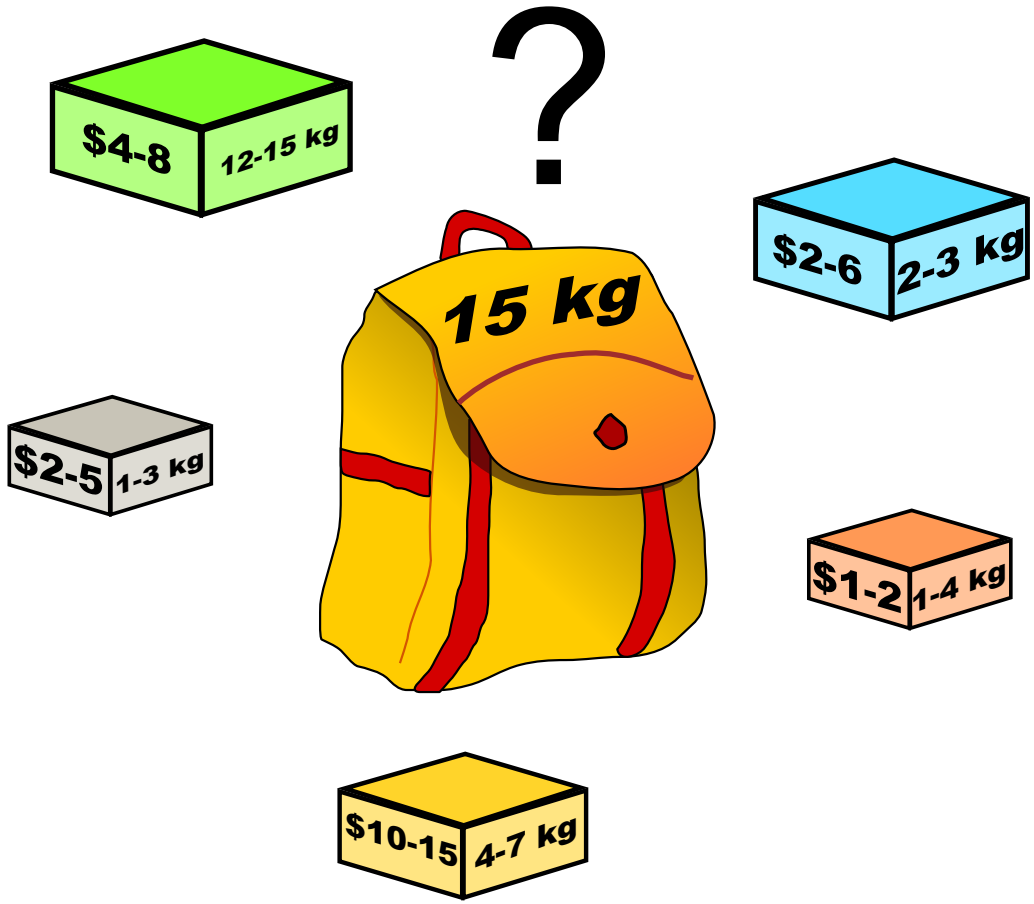
3 Solving Stochastic Combinatorial Optimization problems
- Problems/Difficulties
- Deterministic Reformulation
- Sample Average Approximation
- Metaheuristics for SCO problems

4 Conclusion
- Further Reading

Deterministic Knapsack problem



Stochastic Knapsack problem



Possible ways to handle capacity constraint

- knapsack constraint violated $\Rightarrow$ penalty
- probability of capacity violation restricted
- decision can be corrected later (add. costs/reduced rewards)

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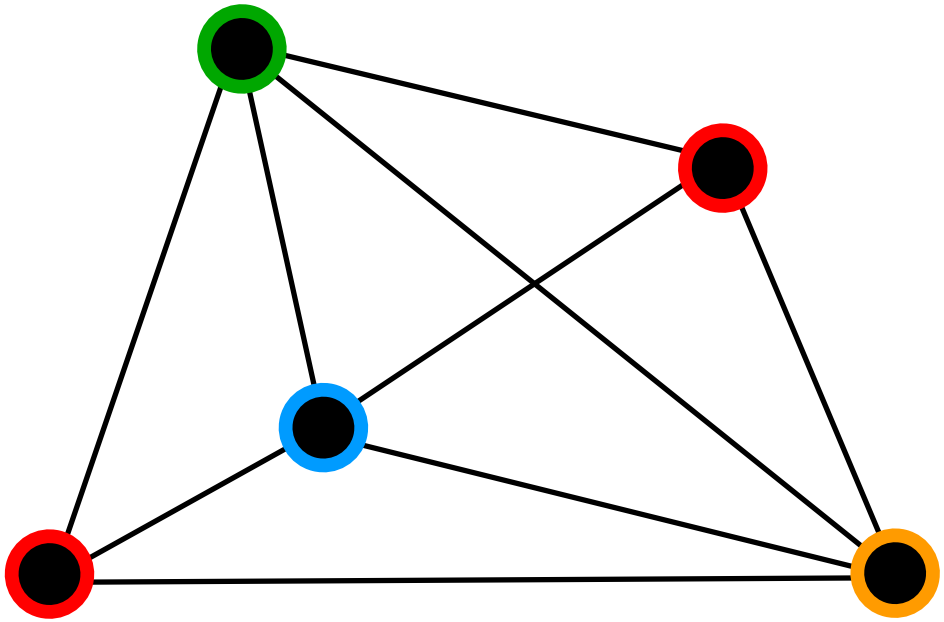
Deterministic Graph Coloring



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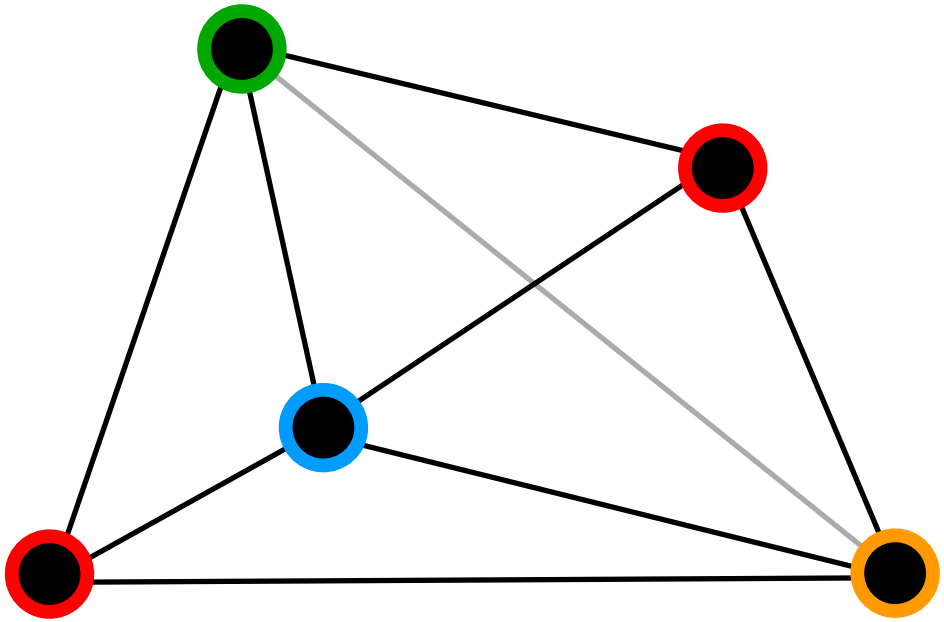

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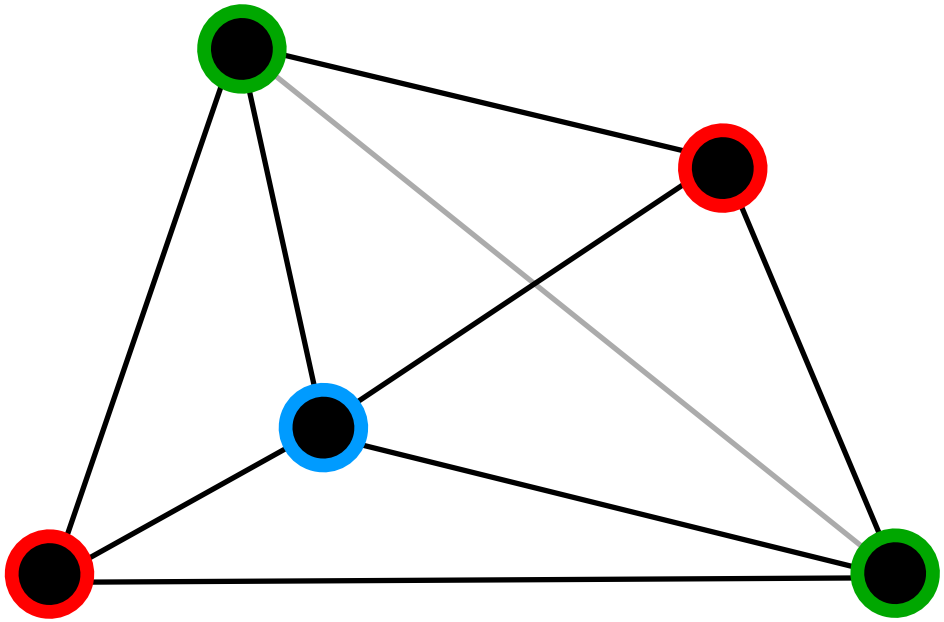

Changing settings

- set of edges random
- set of vertices random

Changing parameters

- allowed number of colors random
- "cost" of colors random

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Deterministic CO Model $\rightarrow$ Stochastic CO Model

$$\begin{array}{lll} \max\limits_{x\in\{0,1\}^n} & f(x) & \\ \text{s.t.} & g(x) \leq 0 & \end{array} \quad \rightarrow \quad \begin{array}{lll} \min\limits_{x\in\{0,1\}^n} & f(x,\chi) & \\ \text{s.t.} & g(x,\chi) \leq 0 & \end{array}$$

$\chi \in \Omega \subseteq \mathbb{R}^s$: vector with random entries

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Minimize an expectation

$$\min_{x \in \{0,1\}^n} \quad \mathbb{E}\left[f(x, \chi)\right]$$
$$\text{s.t.} \quad g(x) \leq 0$$

Advantages:

- Good result "on average"
- Objective function can often be reformulated deterministically

Disadvantages:

- We might encounter very "bad cases"

Minimize variance

$$\min_{x \in \{0,1\}^n} \quad Var\left[f(x, \chi)\right]$$
$$\text{s.t.} \quad g(x) \leq 0$$

Advantages:

- Outcome more concentrated around mean
- Possibility to reduce risk

Disadvantages:

- Makes not much sense without benchmark for expected costs

Minimize variance

$$\min_{x \in \{0,1\}^n} \quad \lambda Var\left[f(x,\chi)\right] + \mathbb{E}\left[f(x,\chi)\right]$$
$$\text{s.t.} \quad g(x) \leq 0$$

Advantages:

- Outcome more concentrated around mean
- Possibility to reduce risk

Minimize variance

$$\min_{x \in \{0,1\}^n} \quad Var\left[f(x,\chi)\right]^{\lambda} \cdot \mathbb{E}\left[f(x,\chi)\right]$$
$$\text{s.t.} \quad g(x) \leq 0$$

Advantages:

- Outcome more concentrated around mean
- Possibility to reduce risk

Robust optimization

$$\min_{x \in \{0,1\}^n} \max_{\chi \in \Omega} \quad f(x, \chi)$$
$$\text{s.t.} \quad g(x) \leq 0$$

Advantages:

- Worst case not too bad: Solution is *robust*

Disadvantages:

- $f(x, \cdot)$ needs to be bounded from above
- Worst case might be very improbable
- Average might be high

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Worst case model

$$\min_{x \in \{0,1\}^n} \quad f(x)$$
$$\text{s.t.} \quad g(x, \chi) \leq 0 \quad \forall \chi \in \Omega$$

Advantages:

- Absolutely robust solution

Disadvantages:

- Problem often infeasible or has only trivial solutions
- Solution at high costs
- Constraint forced to be satisfied in even very improbable cases

Chance-Constrained model

$$\min_{x \in \{0,1\}^n} \quad f(x)$$
$$\text{s.t.} \quad \mathbb{P}\{\exists i : g_i(x, \chi) > 0\} \leq \alpha$$

Advantages:

- Very improbable cases can be ignored
- Cost can be reduced

Disadvantages:

- No restriction of "magnitude" of allowed violation
- What happens if constraint is violated?

Simple-Recourse model

$$\min_{x \in \{0,1\}^{n_1}} \quad f(x) + \sum_{i=1}^{m} d_i \cdot \mathbb{E}\left[[g_i(x, \chi)]^+\right]$$

Advantages:

- Costs in case of violation taken into account
- "Magnitude" of violation can be controlled

Disadvantages:

- Probability of violation not restricted

Two-Stage model

$$\min_{x \in \{0,1\}^{n_1}} \quad f(x) + \mathbb{E}\left[Q(x,\chi)\right]$$

$$\text{s.t.} \quad Q(x,\chi) = \min_{y \in \{0,1\}^{n_2}} \quad \overline{f}(y)$$

$$\text{s.t.} \quad g(x,y,\chi) \leq 0$$

Advantages:

- Violation of constraint not permitted
- Corrections in case of violation taken into account

Disadvantages:

- Problem extremely hard to solve:
  - → Non-convex, non-continuous objective function
  - → No closed-form expression of objective function
  - → Second-stage problem $\mathcal{NP}$-hard

Deterministic Knapsack Problem

$$\max_{x \in \{0,1\}^n} \quad \sum_{i=1}^{n} r_i x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} w_i x_i \leq c$$

Simple Recourse Knapsack Problem

$$\max_{x \in \{0,1\}^n} \sum_{i=1}^{n} r_i x_i - d \cdot \mathbb{E}\left[[\sum_{i=1}^{n} \chi_i x_i - c]^+ \right]$$

Two-Stage Knapsack Problem

$$(TSKP) \quad \max_{x\in\{0,1\}^n} \quad \sum_{i=1}^{n} r_i x_i + \mathbb{E}[\mathcal{Q}(x,\chi)]$$

$$\text{s.t.} \quad \mathcal{Q}(x,\chi) = \max_{y^+,y^-\in\{0,1\}^n} \sum_{i=1}^{n} \bar{r}_i y_i^+ - \sum_{i=1}^{n} d_i y_i^-,$$

$$\text{s.t.} \quad y_j^+ \le 1 - x_j, \quad j = 1,\dots,n,$$

$$y_j^- \le x_j, \quad j = 1,\dots,n,$$

$$\sum_{i=1}^{n} (x_i + y_i^+ - y_i^-)\chi_i \le c.$$

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Static Stochastic Optimization problems

- Random parameters revealed after decision has been made.
- For decision maker parameters are revealed "once for all".
- No corrective decision can be made.

- Minimize expectation and/or variance
- Robust/Worst Case Optimization
- Chance-Constrained Optimization
- Simple Recourse Model

Two-Stage Optimization problems

- For decision maker parameters are revealed "once for all".
- Random parameters revealed after first-stage decision has been made.
- Corrective decision can be made once the parameters are known.

- Two-Stage Model
- Simple Recourse Model

Multi-Stage Optimization problems

- Parameters are revealed in several stages.
- Corrective decision can be made in each stage.
- Which parameters are revealed in which stage generally defined.
- Decisions do only depend on already revealed parameters.

- Two-Stage Model
- Multi-Stage Model

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Structural Difficulties

- Non-convexity
- Non-continuous objective functions
- No closed-form expression for objective function

Two-Stage model

$$\begin{aligned} \min_{x \in \{0,1\}^{n_1}} \quad & f(x) + \mathbb{E}\left[Q(x,\chi)\right] \\ \text{s.t.} \quad & Q(x,\chi) = \min_{y \in \{0,1\}^{n_2}} \quad \overline{f}(y) \\ & \qquad\qquad \text{s.t.} \quad g(x,y,\chi) \leq 0 \end{aligned}$$

Computational Difficulties

- Expectations and probabilities = (multi-dimensional) integrals!
- Evaluating objective function might be $\mathcal{NP}$-hard
- High number of binary decision variables and constraints

Two-Stage Knapsack Problem

$$(TSKP) \quad \max_{x\in\{0,1\}^n} \quad \sum_{i=1}^{n} r_i x_i + \mathbb{E}[\mathcal{Q}(x,\chi)]$$

$$\text{s.t.} \quad \mathcal{Q}(x,\chi) = \max_{y^+,y^-\in\{0,1\}^n} \sum_{i=1}^{n} \bar{r}_i y_i^+ - \sum_{i=1}^{n} d_i y_i^-,$$

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$$y_j^- \le x_j, \quad j = 1,\dots,n,$$

$$\sum_{i=1}^{n} (x_i + y_i^+ - y_i^-)\chi_i \le c.$$

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Idea

→ Reformulate problem as a deterministic optimization problem

→ Use already existing solvers to solve obtained problem

→ Adapt existing algorithms to the special structure of the obtained problem

Problem

Generally only possible under assumption of special distributions!

Ex. 1: Chance-constrained Knapsack prob. (normal distribution)

$$\max_{x \in \{0,1\}^n} \quad \sum_{i=1}^{n} r_i x_i$$

$$\text{s.t.} \quad \mathbb{P}\{\sum_{i=1}^{n} \chi_i x_i > c\} \leq \alpha$$

Assume:

$\chi \sim \mathcal{N}(\mu, \Sigma)$ and $\alpha < 0.5 \Rightarrow$

$$\mathbb{P}\{\sum_{i=1}^{n} \chi_i x_i > c\} \leq \alpha \Leftrightarrow \underbrace{\sum_{i=1}^{n} x_i \mu^i + \Phi^{-1}(1-\alpha)\|\Sigma^{1/2} x\| \leq c}_{\text{Second Order Cone Constraint!}}$$

Ex. 2: Chance-constrained Knapsack prob. (discrete distribution)

$$\max_{x\in\{0,1\}^n} \sum_{i=1}^{n} r_i x_i \qquad \text{s.t.} \qquad \mathbb{P}\{\sum_{i=1}^{n} \chi_i x_i > c\} \leq \alpha$$

Assume:

K outcomes $\chi^1, \ldots, \chi^K$ with prob.'s $p^1, \ldots, p^K$

Introduce:

K binary decision variables $z_1, \ldots, z_K$

Replace Chance-Constraint by

$$\sum_{i=1}^{n} \chi_i^k x_i \leq c + M z_k \quad \forall\, k = 1, \ldots, K,$$

$$\sum_{k=1}^{K} p_k z_k \leq \alpha$$

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Idea of the SAA

$\rightarrow$ Sample K outcomes for random parameters

$\rightarrow$ Assign probability $1/K$ to each sample

$\rightarrow$ Replace distribution by finite sample

Advantages

- *Approximate* information about underlying distribution ✓
- *Approximate* closed-form expression for objective function ✓
- *Approximate* deterministic reformulation of problem ✓
- *Smaller* number of scenarios ✓

Idea of the SAA

→ Sample K outcomes for random parameters

→ Assign probability $1/K$ to each sample

→ Replace distribution by finite sample

Disadvantages

- Solution of SAA might be infeasible for original problem
- Solution of SAA might be non-optimal for original problem
- To approximate original problem high number of samples might be needed

Simple Recourse Knapsack Problem

$$\max_{x\in\{0,1\}^n} \sum_{i=1}^{n} r_i x_i - d \cdot \mathbb{E}\left[\left[\sum_{i=1}^{n} \chi_i x_i - c\right]^+\right]$$

Sample Average Approximation:

$$\max_{x\in\{0,1\}^n} \sum_{i=1}^{n} r_i x_i - d \cdot \sum_{k=1}^{K} \frac{1}{K}\left[\sum_{i=1}^{n} \chi_i^k x_i - c\right]^+$$

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"Positive" Features

- Basically same as for deterministic comb. opt.
- Based on sample average approximations
- Increase sample size to obtain convergence.
- Additional diversification due to randomness of samples

Difficulties

- Evaluation of objective function expensive
- Comparison of quality of solutions difficult
- SCO problems generally have a lot of local optima
- Values of local optima can be close

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Summary

- Modeling SCO problems:
  - → Randomness in objective function?
  - → Randomness in constraint?
  - → Violation of constraint possible? Penalty?
  - → Correction of decision possible? How often?
- Solving SCO problems:
  - → Deterministic equivalent formulation
  - → Approximation of problem using sampling
  - → Meta-Heuristics

Greatest Challenges in Stochastic Combinatorial Optimization

- Modeling Combinatorial Real World problems with uncertainties as SCO problems
- Solve "realistic" sized problems in reasonable time
- Find more adapted solution techniques (for general distributions)
- Find efficient (Meta)Heuristics

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